COOPERATIVE GAME THEORY

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Contents

Chapter 1

Cooperative Games

Bibliography

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- [2] M. Maschler, E. Solan, S. Zamir. *Game Theory*. Cambridge University Press, 2013.

- [3] B. Peleg, P. Sudhölter. *Introduction to the theory of cooperative games*. Kluwer Academic Publisher, 2003.

- [4] H. Peters. *Game Theory: A Multilevel Approach*. Springer, 2008.

Most of the material of this course comes from [3] and [4].

1.1 Introduction

A game can be given in:

- normal form (matrix)
- extensive form (tree)
- characteristic form (function)

The *characteristic form* of a game summarizes the possible gains/losses/utilities for a given group of players (called *coalition*), the group being bound by a cooperation agreement. The details of the strategies to obtain these values are ignored.

Notation:

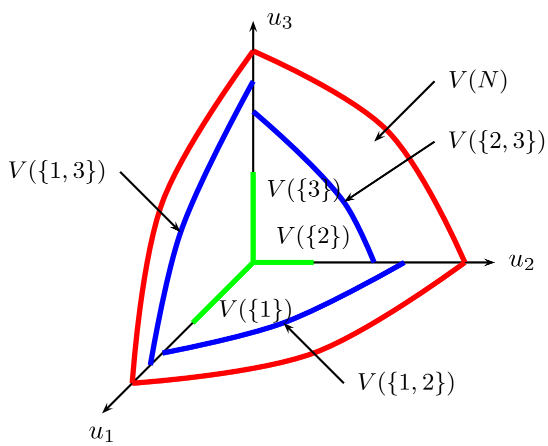
- $N = \{1, \ldots, n\}$: set of players
- $S \subseteq N$: coalition. N is called the *grand coalition*.
- $u = (u_1, \ldots, u_n) \in \mathbb{R}^N$: outcome, i.e., utility vector for each player. We denote by $u_S \in \mathbb{R}^S$ the restriction of u to S.

Definition 1.1. A *set of feasible utilities* is a nonempty closed set $U_S \subseteq \mathbb{R}^S$ which is *comprehensive*, i.e., it satisfies

$$u_S \in U_S, \quad u'_S \leqslant u_S \Rightarrow u'_S \in U_S.$$

Definition 1.2. A game in *characteristic form* is a pair (N, V) where N is the set of players and V is a function from 2^N to $2^{\mathbb{R}^N}$, $S \mapsto V(S) \subseteq \mathbb{R}^S$, with $V(S)$ the set of possible utilities for S when the players of S cooperate.

Example 1.1. Example with $n = 3$.

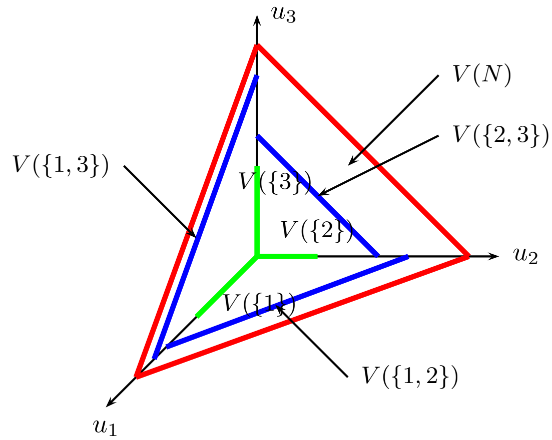


An important particular case is when utilities are *transferable* between players in S. Supposing that after normalization utilities have the same unit for each player, the general form of $V(S)$ is a half-plane of the form:

$$V(S) = \{u_S \in \mathbb{R}^S \mid \sum_{i \in S} u_i \leqslant v(S)\}$$

with $v(S) \in \mathbb{R}$, the maximal amount of utility the players of S can obtain and freely share among them, by transfer between players called *side payments*.

Example 1.2. The preceding example with $n = 3$ becomes, with transferable utilities:



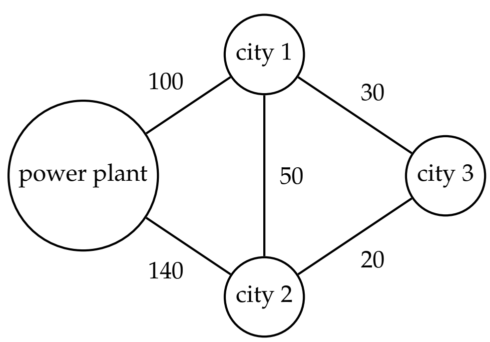
Let us remark that in the case of transferable utilities, the game is completely characterized by the quantities $v(S)$, $S \subseteq N$. This kind of game is called *game with transferable utilities under characteristic form*, or *TU-game* for short, and denoted by (N, v) with $v : 2^N \to \mathbb{R}$, with $v(\varnothing) := 0$.

The other games under characteristic form are called *NTU* (non-transferable utilities).

1.2 Examples

1.2.1 The three cities

Consider three cities in an electricity network with connection costs as follows:



If the cities cooperate, they can save connection costs. Let $N = \{1, 2, 3\}$ be the set of the 3 cities. For a coalition $S \subseteq N$ of cities, let $c(S)$ be the minimal cost of connection for the cities in S, and $v(S)$ the benefit of cooperation

$$v(S) := \sum_{i \in S} c(\{i\}) - c(S)$$

We have

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

1.2.2 The glove game

Consider 3 players. Players 1 and 2 possess a right glove, and player 3 a left glove. A pair of gloves has worth 1. The players must cooperate for generating profit.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	1	1	1

1.2.3 Permutation games

Abel, Banach and Cantor have an appointment at the dentist on Monday, Tuesday and Wednesday, respectively. However, these appointments are not completely satisfactory for them. Their preference expressed as a utility are:

	Monday	Tuesday	Wednesday
Abel	2	4	8
Banach	10	5	2
Cantor	10	6	4

If they cooperate, they can exchange their appointments and arrive at a better overall utility (expressed by v, to be compared with u, the initial total utility):

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	N
$v(S)$	2	5	4	14	18	9	24
$u(S)$	2	5	4	7	6	9	11

1.2.4 Airport games

The problem is to compute airport taxes for planes. There are two types of cost:

- variable cost for landing operations
- fixed cost (e.g., construction of a new landing track, a new terminal, etc.)

Variable costs are directly assigned to landing planes. Fixed costs must be shared among planes (depending on their type).

We consider m types of planes, with N_j the set of planes of type j, and $N = \bigcup_{j=1}^m N_j$. We define c_j the cost of a landing track suitable for type j, and $c(S)$ is the cost of a landing track suitable for all planes in $S \subseteq N$:

$$c(S) = \max\{c_j \mid S \cap N_j \neq \varnothing\}, \quad c(\varnothing) = 0.$$

1.2.5 Bankruptcy games

Problem extracted from the Talmud, by Rabbi Ibn Ezra (1140):

> *Jacob died, and his 4 sons Reuben, Simon, Lévi and Judas produced each of them a will saying that Jacob give respectively all his estate, half of it, a third of it, and a fourth of it. His estate is valued to 120 units.*

A bankruptcy problem is a pair (E, d), where $E \in \mathbb{R}_+$ is the estate and $d \in \mathbb{R}^n_+$ is the vector of debts. The associated bankruptcy game is $(N, v_{E,d})$, with $N = \{1, \ldots, n\}$ the set of claimants, and

$$v_{E,d}(S) = \max(0, E - \sum_{j \in N \setminus S} d_j) \qquad (S \subseteq N).$$

Let us apply this to Jacob's problem:

$$d_1 = 120, \quad d_2 = 60, \quad d_3 = 40, \quad d_4 = 30$$

hence (omitting commas and braces, i.e., writing 123 instead of $\{1, 2, 3\}$):

$$\begin{gathered} v(N) = 120 \\ v(1) = v(2) = v(3) = v(4) = 0 \\ v(12) = 50,\ v(13) = 30,\ v(14) = 20,\ v(23) = v(24) = v(34) = 0 \\ v(123) = 90,\ v(124) = 80,\ v(134) = 60,\ v(234) = 0. \end{gathered}$$

1.3 Cooperative TU-games

Remark: Cooperative games are also called *coalitional games*.

1.3.1 Definitions

We set $G = (N, v)$.

Definition 1.3. (i) A game (N, v) is *superadditive* if

$$v(S \cup T) \geqslant v(S) + v(T) \qquad (S, T \text{ disjoint})$$

It is *subadditive* if the reverse inequality holds.

(ii) A game (N, v) is *convex* if

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T) \qquad (S, T \in 2^N)$$

It is *concave* if the reverse inequality holds.

Some remarks:

(i) Superadditivity is a natural property if v is a profit, a benefit or a utility (cooperation produces more profit). Subadditivity is natural if v is a cost.

(ii) A game which is both superadditive and subadditive is called *additive*. Note that a game is additive iff it is both convex and concave.

(iii) Convexity implies superadditivity: it is a stronger property.

(iv) An equivalent definition of convexity is:

$$v(S \cup \{i\}) - v(S \leqslant v(T \cup \{i\}) - v(T), \qquad (i \in N)(S \subseteq T \subseteq N \setminus \{i\}). \quad (1.1)$$

Interpretation of (1.1): the marginal contribution of i is increasing with the size of the coalition.

Exercise 1.1. Prove the equivalence of (1.1) with convexity.

Definition 1.4. (i) A game (N, v) is a *constant sum game* if

$$v(S) + v(N \setminus S) = v(N) \qquad (S \in 2^N).$$

(ii) A game (N, v) is *inessential* if it is additive:

$$v(S) = \sum_{i \in S} v(\{i\}) \qquad (S \in 2^N).$$

An inessential game has no interest since if every player i claims a payment at least equal to $v(\{i\})$, then the only possible payment is $x_i = v(\{i\})$ for all $i \in N$.

Clearly, an additive game is equivalent to a vector $x \in \mathbb{R}^N$, hence the notation

$$x(S) := \sum_{i \in S} x_i$$

for any $S \subseteq N$ (by convention, $x(\varnothing) = 0$) which will be very often used.

Definition 1.5. Two games (N, v) and (N, w) are *strategically equivalent* if there exists $\alpha > 0, \beta \in \mathbb{R}^N$ such that

$$w(S) = \alpha v(S) + \underbrace{\beta(S)}_{\sum_{i \in S} \beta_i}$$

When v, w are two strategically equivalent games, w can be deduced from v by a change of unit (α) and different initial endowments (β) to the players.

Definition 1.6. (i) A game (N, v) is *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$.

(ii) A game (N, v) is *monotone* if $S \subseteq T$ implies $v(S) \leqslant v(T)$.

(iii) A game (N, v) is *symmetric* if for all permutation π on N, we have $v(\pi(S)) = v(S)$ for all $S \in 2^N$ (equivalently: if $v(S) = v(T)$ whenever $|S| = |T|$, for all $S, T \in 2^N$).

Exercise 1.2. (i) Prove that any game (N, v) is strategically equivalent to a zero-normalized game.

(ii) Prove that any game (N, v) is strategically equivalent to a monotone game.

1.3.2 Cost allocation games

Let $c : 2^N \to \mathbb{R}$ be a cost function on N (set of clients), where $c(S)$ is the minimal cost for serving the clients in $S \subseteq N$. To (N, c) we associate a TU game (N, v) representing the realized saving:

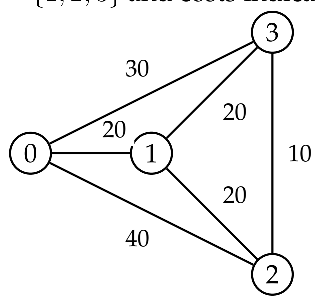
$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) \qquad (S \in 2^N).$$

(N, c) is subadditive (resp., concave) iff (N, v) is superadditive (resp., convex) (why?).

Examples of cost allocation games are the three cities game, the airport game, and the minimal cost spanning tree game (general form of the 3 cities game), presented below.

A *minimal cost spanning tree game* is a pair (N, c) where N is a set of clients to be served by a central node denoted by 0 (power plant, water tower, etc.). We put $N_* = N \cup \{0\}$ and consider the complete graph on N_*, with cost c_{ij} on the link between i and j. For every subset $S \subseteq N$, we define $c(S)$ to be the minimal cost of all spanning trees over $S \cup \{0\}$, where the cost of a tree is defined as the sum of the costs of all links in the tree.

Example 1.3. Consider $N = \{1, 2, 3\}$ and costs indicated on the graph below.



Then

$$c(1) = 20, \quad c(3) = 30, \quad c(123) = 50, \quad c(S) = 40 \text{ otherwise.}$$

1.3.3 Simple games

Definition 1.7. A *simple game* is a pair $(N, \mathcal{W})$ where $\mathcal{W} \subseteq 2^N$ is the set of *winning coalitions*, satisfying

(i) $N \in \mathcal{W}$

(ii) $\varnothing \notin \mathcal{W}$

(iii) $S \subseteq T$, $S \in \mathcal{W}$ imply $T \in \mathcal{W}$.

Equivalently, a simple game $g = (N, \mathcal{W})$ can be represented by a pair $G = (N, v)$, where $v : 2^N \to \{0, 1\}$ is the associated coalitional game, defined by

$$v(S) = 1 \Leftrightarrow S \in \mathcal{W}.$$

Note that a simple game is monotone (why?). Simple games are typically applied to votes in a committee N. Winning coalitions are those which have the decision power: every voter votes 'yes' or 'no', and the final decision is 'yes' iff $S = \{\text{voters voting 'yes'}\} \in \mathcal{W}$.

Definition 1.8. A winning coalition is *minimal* if every proper subset of it is not winning. We denote by

$$\mathcal{W}^m = \{S \in \mathcal{W} \mid T \subset S \Rightarrow T \notin \mathcal{W}\}$$

the set of minimal winning coalitions.

Remarks:

(i) $\mathcal{W}^m$ completely determines the simple game.

(ii) $\mathcal{W}^m$ is an antichain in the Boolean lattice $(2^N, \subseteq)$. Conversely, any antichain, except $\{\varnothing\}$, determines a simple game. It follows that the number of simple games is the number of antichains on 2^N minus 1. The number of antichains of 2^N is the *Dedekind*[1] *number* $M(n)$, which is known only till $n = 8$ (see the table below).

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

Table 1.1: The Dedekind numbers $M(n)$ for $0 \leqslant n \leqslant 8$

Definition 1.9. Let $g = (N, \mathcal{W})$ be a simple game. The game is

(i) *proper* if $S \in \mathcal{W} \Rightarrow N \setminus S \notin \mathcal{W}$

(ii) *strong* if $S \notin \mathcal{W} \Leftrightarrow N \setminus S \in \mathcal{W}$

(iii) *weak* if $V = \bigcap_{S \in \mathcal{W}} \neq \varnothing$. The members of V are the *veto players*.

(iv) *dictatorial* if there is exactly one veto player.

Exercise 1.3. Prove that, if g is a simple game with G the associated coalitional game,

(i) G superadditive iff g proper

(ii) G constant sum iff g strong.

Definition 1.10. A simple game $g = (N, \mathcal{W})$ is a *weighted majority game* if there exist $q > 0$ (called the *quota*) and $w_i \geqslant 0$, $i \in N$ (called the *weights*) such that $S \in \mathcal{W}$ iff $w(S) \geqslant q$.

Notation: $g = (q; w_1, \ldots, w_n)$.

Example 1.4. Let $g = (39; 7, 7, 7, 7, 7, \underbrace{1, \ldots, 1}_{10 \text{ times}})$. Then g is weak and the veto players are the five first ones.

[1]Richard Dedekind (Braunschweig, 1831 – Braunschweig, 1916), German mathematician.

Definition 1.11. Let $T \subseteq N$, $T \neq \varnothing$. The *unanimity game centered on* T is a game defined:

$$u_T(S) = \begin{cases} 1, & \text{if } S \supseteq T \\ 0, & \text{otherwise.} \end{cases}$$

Note that u_T is a simple game with $\mathcal{W}^m = \{T\}$. Dictatorial games are unanimity games $u_{\{i\}}$, $i \in N$, and u_T is a weak simple game whose veto players are the players in T.

Chapter 2

Solution concepts and the core

Throughout the chapter, we consider a TU-game (N, v).

2.1 Solution concept

Supposing that the grand coalition N forms, it remains to solve the problem of sharing the benefit $v(N)$ among the players in N, in a rational and equitable way. A *solution* (of the game) is a systematic way of sharing $v(N)$, for every game (N, v).

Definition 2.1. We denote by $X(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leqslant v(N)\}$ the set of feasible payments of the game (N, v).

Definition 2.2. Let Γ be a set of games on N. A solution is a mapping

$$\begin{array}{rccl} \sigma: & \Gamma & \rightarrow & 2^{X(N,v)} \\ & (N,v) & \mapsto & \sigma(N,v) \subseteq X(N,v). \end{array}$$

We distinguish between "point-type" solutions ($\sigma(N, v)$ is a singleton) and "set-type" solutions ($\sigma(N, v)$ is a subset of feasible payments).

Definition 2.3. Let $x \in X(N, v)$.

(i) x is *efficient* if $x(N) = v(N)$.

(ii) x is *individually rational* if $x_i \geqslant v(\{i\})$ for all $i \in N$.

(iii) x is *coalitionally rational* if $x(S) \geqslant v(S)$ for all $S \subseteq N$.

Based on these properties, we define

(i) The set of *pre-imputations* $PI(N, v) = \{x \in X(N, v) \mid x \text{ is efficient}\}$

(ii) The set of *imputations* $I(N, v) = \{x \in PI(N, v) \mid x \text{ is individually rational}\}$

If there is no ambiguity, we may use the notation $X(v), PI(v), I(v)$ for simplicity. Two remarks:

- If x is not efficient, a portion of $v(N)$ is wasted.

- If x is not individually rational, the players i such that $x_i < v(\{i\})$ have no interest to participate to the game.

Proposition 2.1. (i) $I(v) \neq \varnothing$ iff $v(N) \geqslant \sum_{i \in N} v(\{i\})$

(ii) $I(v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$ if v additive.

(the proof is left to the reader as an exercise; is the converse of (ii) true?)

Definition 2.4. The *core* of (N, v) is the set

$$C(N,v) = \{x \in \mathbb{R}^N \mid x(S) \geqslant v(S), \forall S \subseteq N,\ x(N) = v(N)\}$$

of coalitionally rational (pre-)imputations.

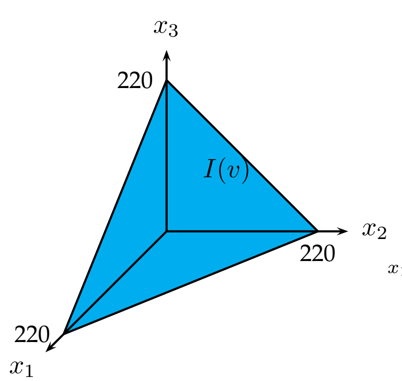
Note that if x is not coalitionally rational, coalitions S such that $x(S) < v(S)$ can leave the grand coalition N.

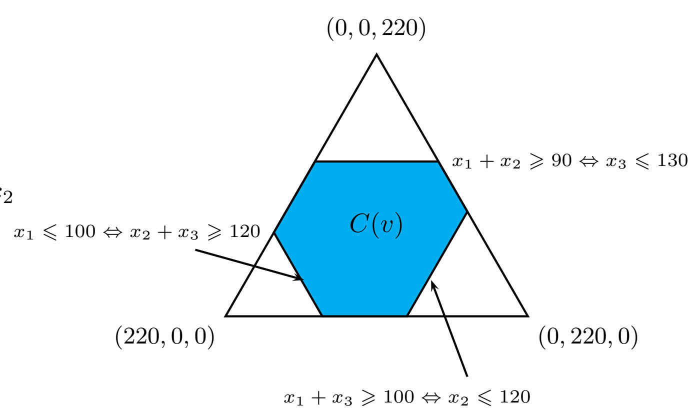
Example 2.1. We consider the 3 cities game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	90	100	120	220

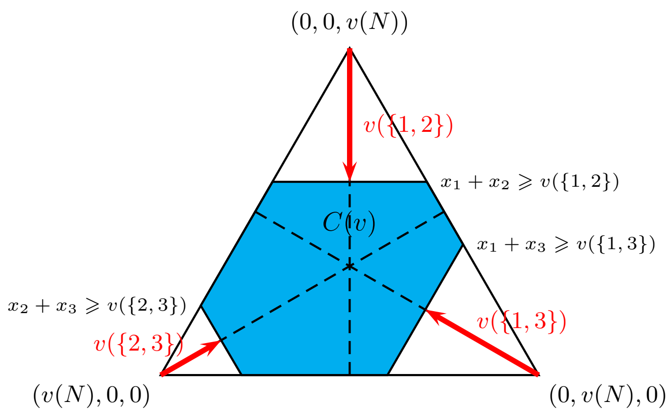
We have:

$$\begin{aligned}
PI(v) &= \{x \in \mathbb{R}^3 \mid & x_1 + x_2 + x_3 = 220\} \\
I(v) &= \{x \in \mathbb{R}^3_+ \mid & x_1 + x_2 + x_3 = 220\} \\
C(v) &= \{x \in \mathbb{R}^3_+ \mid & x_1 + x_2 \geqslant 90 \\
& & x_1 + x_3 \geqslant 100 \\
& & x_2 + x_3 \geqslant 120 \\
& & x_1 + x_2 + x_3 = 220\}
\end{aligned}$$





For a game with 3 players, it is convenient to represent the core by the so-called "triangle diagram", provided the game is zero-normalized. The general form is given below.



Generally speaking, the core is bounded convex closed polyhedron, possibly empty, of dimension at most $n - 1$. Therefore, it is either of infinite cardinality, or reduced to a singleton, or empty.

2.2 Domination and stable sets

Definition 2.5. (Von Neumann and Morgenstern, 1944) Let (N, v) be a game, and $y, z \in I(N, v)$, $S \subseteq N$, $S \neq \varnothing$. We say that y *dominates* z *via coalition* S, denoted by $y \operatorname{dom}_S z$, if

(i) $y_i > z_i$ for all $i \in S$

(ii) $y(S) \leqslant v(S)$.

y *dominates* z, denoted by $y \operatorname{dom} z$, if there exists a nonempty set $S \subseteq N$ such that $y \operatorname{dom}_S z$.

Condition (i) can be interpreted as: the payment y is strictly better than z for all members of S, and as for (ii): the payment y is affordable by cooperation of the members of S.

We introduce

$$D(S) = \{z \in I(v) \mid \exists y \in I(v), y \operatorname{dom}_S z\}$$

the set of payments dominated via S. Note that $D(N) = \varnothing$ and $D(\{i\}) = \varnothing$, $\forall i \in N$ (why?). Observe that if $z \in D(S)$, then the players in S can raise a valid objection against z.

Definition 2.6. An imputation $x \in I(v)$ is *undominated* if $x \in I(V) \setminus \bigcup_{S \in 2^N, S \neq \varnothing} D(S)$.

Example 2.2. Let (N, v) with $N = \{1, 2, 3\}$ and v defined by

$$v(\{1, 2\}) = 2, \quad v(N) = 1, \quad v(S) = 0 \text{ otherwise.}$$

Then $D(S) = \varnothing$ if $S \neq \{1, 2\}$ and $D(\{1, 2\}) = \{x \in I(v) \mid x_3 > 0\}$. Therefore, x is undominated if $x_3 = 0$.

Exercise 2.1. Prove the above statements in the example.

Definition 2.7. The *domination core* of a game (N, v) is the set

$$DC(v) = I(v) \setminus \bigcup_{S \in 2^N, S \neq \varnothing} D(S)$$

of undominated elements of $I(v)$.

In Ex. 2.2, the core is empty, but the domination core is not empty (check it!). A general statement about the core and domination core is the following.

Theorem 2.1. $C(N, v) \subseteq DC(N, v)$ for every game (N, v).

Proof. Let $x \in I(v)$ and $x \notin DC(v)$. Then $\exists y \in I(v)$, $\exists S \in 2^N$ such that $y \operatorname{dom}_S x$. Then $v(S) \geqslant y(S) > x(S)$, which implies $x \notin C(v)$. □

Theorem 2.2. Let (N, v) be a game. We have $DC(N, v) = C(N, v)$ if

$$v(N) \geqslant v(S) + \sum_{i \in N \setminus S} v(\{i\}) \qquad (S \in 2^N \setminus \{\varnothing\}). \tag{2.1}$$

Proof. It remains to show that $DC(v) \subseteq C(v)$.

1. Let us show that $x \in I(v)$ with $x(S) < v(S)$ for a given S implies that there exists $y \in I(v)$ s.t. $y \operatorname{dom}_S x$. Let us define y as follows:

- If $i \in S$, $y_i := x_i + \frac{1}{|S|}(v(S) - x(S))$.
- If $i \notin S$, then $y_i = v(\{i\}) + \frac{1}{|N \setminus S|}(v(N) - v(S) - \sum_{j \in N \setminus S} v(\{j\}))$

Then $y_i \geqslant v(\{i\})$ for all $i \in N \setminus S$ by (2.1), and it can be checked that $y \in I(v)$. Moreover, $y \operatorname{dom}_S x$.

2. Let $x \in DC(v)$. Then there is no $y \in I(v)$ s.t. $y \operatorname{dom} x$. Consequently, $x(S) \geqslant v(S)$ for all $S \in 2^N \setminus \{\varnothing\}$, which proves $x \in C(v)$. □

Observe that if v is super-additive, then it satisfies (2.1). More importantly:

Corollary 2.1. Let (N, v) be a game. If $C(N, v) \neq \varnothing$, then $DC(N, v) = C(N, v)$.

Proof. Suppose that $C(N, v) \neq \emptyset$ and pick $x \in C(N, v)$. Then for any nonempty $S \in 2^N$,

$$v(N) = x(N) = x(S) + \sum_{i \in N \setminus S} x_i \geqslant v(S) + \sum_{i \in N \setminus S} v(\{i\}).$$

Using Theorem 2.2, we deduce that $DC(N, v) = C(Nv)$. □

Definition 2.8. (Von Neumann and Morgenstern, 1944) Let (N, v) be a game and $A \subseteq I(N, v)$. A is a *stable set* if

- **internal stability:** If $x, y \in A$ then x does not dominate y
- **external stability:** If $x \in I(N, v) \setminus A$, then there exists $y \in A$ which dominates x.

Example 2.3. Consider $N = \{1,2,3\}$ and v given by $v(S) = 1$ if $|S| > 1$, $v(S) = 0$ otherwise. Then

$$A = \Big\{\Big(\frac{1}{2},\frac{1}{2},0\Big),\Big(\frac{1}{2},0,\frac{1}{2}\Big),\Big(0,\frac{1}{2},\frac{1}{2}\Big)\Big\}$$

is a stable set. Indeed:

- internal stability: clear.
- external stability: an imputation has the form $(x, y, 1-x-y)$ with the condition $0 \leqslant x + y \leqslant 1$. Observe that at most one component of x is strictly greater than $1/2$, in which case the others are strictly less than $1/2$. Suppose $x_1 > 1/2$. Then x is dominated by $(0, \frac{1}{2}, \frac{1}{2})$ via $\{2,3\}$.

Exercise 2.2. Consider the same game as in Example 2.3. Prove that

$$B_c := \{x \in I(v) \mid x_3 = c\}$$

with $c < \frac{1}{2}$ is a stable set.

Some remarks:

(i) There is no known necessary and sufficient condition on (N, v) for the existence of a stable set.

(ii) As shown in Example 2.3 and Exercise 2.2, it may exist several stable sets, which are not necessarily convex.

For these reasons, stable sets are not convenient as a solution concept.

Some other results (proof is left as an exercise)

Theorem 2.3. Let (N, v) be a simple game and S be a minimal winning coalition. Consider

$$\Delta^S = \{x \in I(v) \mid x_i = 0, \forall i \notin S\}.$$

Then, if $\Delta^S \neq \varnothing$, it is a stable set.

Theorem 2.4. Let (N, v) be a game. Then:

(i) The domination core is included in any stable set.

(ii) If $DC(v)$ is stable, then it is the unique stable set.

2.3 The case of simple games

It can be checked that (proof is left as an exercise)

(i) For a dictatorial game $u_{\{i\}}$,

$$I(u_{\{i\}}) = \{e^i\}, \quad C(u_{\{i\}}) = DC(u_{\{i\}}) = \{e^i\}$$

with $e^i := (0,\dots,0,\underbrace{1}_{\text{position } i},0,\dots,0)$.

(ii) For a majority game with 3 players ($\mathcal{W} = \{S : |S| > 1\}$),

$$C(v) = DC(v) = \varnothing$$

(iii) For a unanimity game u_T:

$$C(u_T) = DC(u_T) = \operatorname{conv}\{e^i, i \in T\}.$$

Theorem 2.5. Let (N, v) be a simple game. Then

(i) $C(v) = \operatorname{conv}(e^i \mid i \in \operatorname{veto}(v)\}$, where $\operatorname{veto}(v)$ is the set of all veto players of v.

(ii) If $\operatorname{veto}(v) = \varnothing$ and $\{i \in N \mid v(\{i\}) = 1\} = \{k\}$, then $C(v) = \varnothing$ and $DC(v) = \{e^k\}$. Otherwise, $DC(v) = C(v)$.

Proof. 1. Let $i \in \operatorname{veto}(v)$ and $S \subseteq N$, $S \neq \varnothing$. Then $e^i \in C(v)$ since $i \in S$ implies $e^i(S) = 1 \geqslant v(S)$, otherwise $e^i(S) = 0 = v(S)$. Also $e^i(N) = 1 = v(N)$. This proves $\supseteq$ in (i).

2. Let us show $\subseteq$ in (i). Let $x \in C(v)$. It suffices to show that $i \notin \operatorname{veto}(v)$ implies $x_i = 0$. Suppose on the contrary $x_i > 0$ for a non-veto player i. Take S s.t. $v(S) = 1$ and $i \notin S$. Then $x(S) = x(N) - x(N \setminus S) \leqslant 1 - x_i < 1$, a contradiction.

3. Let $\operatorname{veto}(v) = \varnothing$ and let k be the only player in $\{i \in N \mid v(\{i\}) = 1$. Then $C(v) = \varnothing$ by (i), but $I(v) = \{e^k\}$. Hence $DC(v) = \{e^k\}$.

If $\operatorname{veto}(v) = \varnothing$ and $\{i \in N \mid v(\{i\}) = 1\} = \varnothing$, then (2.1) is satisfied and thus $C(v) = DC(v)$.

If $\operatorname{veto}(v) = \varnothing$ and $|\{i \in N \mid v(\{i\}) = 1\}| \geqslant 2$, then $I(v) = \varnothing$ and thus $C(v) = DC(v) = \varnothing$.

4. Suppose $\operatorname{veto}(v) \neq \varnothing$. Then $|\{i \in N \mid v(\{i\}) = 1\}| \leqslant 1$. If $\{i \in N \mid v(\{i\}) = 1\} = \{k\}$ then $\operatorname{veto}(v) = \{k\}$ and $I(v) = \{e^k\} = C(v) = DC(v)$.

If $\{i \in N \mid v(\{i\}) = 1\} = \varnothing$, then (2.1) is satisfied and $C(v) = DC(v)$.

□

2.4 Balanced games

Definition 2.9. A collection $\mathcal{B}$ of nonempty subsets of N is *balanced* if there exist $\lambda_S > 0$, $S \in \mathcal{B}$, such that

$$\sum_{S \in \mathcal{B}} \lambda_S 1_S = 1_N.$$

The λ_S are called the *balancing weights*.

Example 2.4. Take $N = \{1, 2, 3\}$ and $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then $\mathcal{B}$ is a balanced collection with balancing weights $1/2$, $1/2$, $1/2$, respectively.

Some remarks:

(i) In general the balancing weights are not unique.

(ii) Every partition of N is a balanced collection with weights equal to 1.

(iii) Interpretation: every player has one unit of time (or energy, etc.), which he can distribute on the different coalitions to which he belongs to. The distribution is balanced if the unit of time is entirely spent by each player.

Definition 2.10. A game (N, v) is *balanced* if for every balanced collection $\mathcal{B}$ it holds
$$v(N) \geqslant \sum_{S \in \mathcal{B}} \lambda_S v(S).$$

Interpretation: the productivity $v(N)$ of the grand coalition N during one unit of time is greater than the total productivity when N is divided in smaller groups.

Theorem 2.6. (Bondareva-Shapley, weak form) Let (N, v) be a game. Then $C(v) \neq \varnothing$ iff v is balanced.

Proof. The proof is based on LP duality. Consider the LP problem (P):

$$\begin{aligned} &\text{Minimize } z = x(N) \\ &\qquad \text{s.t. } x(S) \geqslant v(S), \quad S \in 2^N \setminus \{\varnothing\}. \end{aligned}$$

Observe that (P) is bounded, and $C(v) \neq \varnothing$ iff (P) has an optimal solution with $z^* = v(N)$. The dual program (D) of (P) reads:

$$\begin{aligned} \text{Maximize } w = &\sum_{\substack{S \subseteq N \\ S \neq \varnothing}} \lambda_S v(S) \\ \text{s.t. } &\sum_{\substack{S \subseteq N \\ S \ni i}} \lambda_S = 1, \quad i \in N \\ &\lambda_S \geqslant 0, \quad S \in 2^N \setminus \{\varnothing\} \end{aligned}$$

(D) has a feasible solution: $\lambda_S = 0$ for all $S \neq N$, and $\lambda_N = 1$, which yields $w = v(N)$.

Hence (D) has an optimal solution (as (P) is bounded), and so has (P) by the Duality Theorem, with $w^* = z^* \geq v(N)$.

Therefore, $C(v) \neq \varnothing$ iff $w^* = v(N)$, which happens iff every feasible solution of (D) satisfies
$$\sum_{\substack{S \subseteq N \\ S \neq \varnothing}} \lambda_S v(S) \leqslant v(N),$$
i.e., v is balanced. □

We say that a balanced collection is *minimal* if it does not contain a proper subcollection that is balanced.

Lemma 2.1. A balanced collection is minimal if and only if it has a unique system of balancing weights.

Proof. $\Leftarrow$) Suppose that $\mathcal{B}$ is not minimal. Then there exists $\mathcal{B}^* \subset \mathcal{B}$ that is balanced with a system of balancing weights $(\lambda^*_A)_{A\in\mathcal{B}^*}$. Then $\mathcal{B}$ has infinitely many systems of balancing weights $(\lambda^\alpha_A)_{A\in\mathcal{B}}$, defined by

$$\lambda^\alpha_A = \begin{cases} \alpha\lambda_A + (1-\alpha)\lambda^*_A, & \text{if } A \in \mathcal{B}^* \\ \alpha\lambda_A, & \text{if } A \in \mathcal{B}\setminus\mathcal{B}^* \end{cases}$$

with $0 < \alpha \leqslant 1$.

$\Rightarrow$) Suppose that $\mathcal{B}$ has two different systems of balancing weights $(\lambda_A)_{A\in\mathcal{B}}$ and $(\lambda'_A)_{A\in\mathcal{B}}$. Then there exists $A \in \mathcal{B}$ such that $\lambda'_A > \lambda_A$, and we put

$$\tau = \min\Big\{\frac{\lambda_A}{\lambda'_A - \lambda_A} \mid \lambda'_A > \lambda_A\Big\}.$$

We define the system of weights $(\tilde{\lambda}_A)_{A\in\mathcal{B}}$:

$$\tilde{\lambda}_A = (1+\tau)\lambda_A - \tau\lambda'_A \qquad (A \in \mathcal{B}).$$

Then $\mathcal{B}^* = \{A \in \mathcal{B} \mid \tilde{\lambda}_A > 0\}$ is a proper subcollection of $\mathcal{B}$ that is balanced with system of balancing weights $(\tilde{\lambda}_A)_{A\in\mathcal{B}^*}$. □

Exercise 2.3. Completion of the above proof:

1. In the $\Leftarrow$) part, prove in detail that $(\lambda^\alpha_A)_{A\in\mathcal{B}}$ is a system of balancing weights for $\mathcal{B}$.
2. In the $\Rightarrow$) part, prove in detail that $(\tilde{\lambda}_A)_{A\in\mathcal{B}^*}$ is a system of balancing weights for $\mathcal{B}^*$.

Let us consider the convex polytope

$$F = \Big\{\lambda \in \mathbb{R}^{2^N\setminus\{\varnothing\}} \mid \sum_{\varnothing\neq A\subseteq N} \lambda_A 1_A = 1_N, \quad \lambda_A \geqslant 0, \quad \forall \varnothing \neq A \subseteq N\Big\}.$$

Lemma 2.2. Let $\lambda \in F$ and consider $\mathcal{B} = \{A \subseteq N \mid \lambda_A > 0\}$. Then λ is an extreme point of F if and only if $\mathcal{B}$ is a minimal balanced collection.

Proof. $\Rightarrow$) If $\mathcal{B}$ is not minimal, then there exists $\mathcal{B}^* \subset \mathcal{B}$ that is balanced, with a system of balancing weights $(\lambda^*_A)_{A\in\mathcal{B}^*}$. We set

$$\gamma_A = (1-t)\lambda_A + t\lambda^*_A$$
$$\gamma'_A = (1+t)\lambda_A - t\lambda^*_A$$

for all $A \in \mathcal{B}$, letting $\lambda^*_A = 0$ if $A \notin \mathcal{B}^*$, with $t > 0$ small enough to ensure $\gamma_A, \gamma'_A > 0$ for all $A \in \mathcal{B}$. Then $(\gamma_A)_{A\in\mathcal{B}}$, $(\gamma')_{A\in\mathcal{B}}$ are systems of balancing weights for $\mathcal{B}$ that are different, because $\gamma_A < \gamma'_A$ for all $A \in \mathcal{B}\setminus\mathcal{B}^*$. Moreover, $\lambda_A = \frac{1}{2}(\gamma_A + \gamma'_A)$ for all $A \in \mathcal{B}$, hence λ is not an extreme point.

$\Leftarrow$) Suppose that $\mathcal{B}$ is a minimal collection. If λ is not an extreme point, there exist distinct $\gamma, \gamma' \in F$ such that

$$\lambda_A = \frac{1}{2}(\gamma_A + \gamma'_A) \qquad (A \in 2^N \setminus \{\varnothing\}).$$

Since γ, γ' are nonnegative, $\lambda_A = 0$ implies $\gamma_A = \gamma'_A = 0$, therefore γ, γ' define distinct systems of balancing weights for collections $\mathcal{C}, \mathcal{C}'$, subcollections of $\mathcal{B}$, which by Lemma 2.1 contradicts the minimality of $\mathcal{B}$. □

Corollary 2.2. A minimal balanced collection contains at most n sets.

Proof. From Lemma 2.2, $\mathcal{B}$ is minimal if and only if its unique system of balancing weight corresponds to an extreme point λ of F. Therefore λ is the (unique) solution of a system of at least $2^n - 1$ equalities among the system $\{\sum_{A\ni i}\lambda_A = 1, i \in N; \lambda_A \geqslant 0, A \in 2^N \setminus \{\varnothing\}\}$. Since the number of equalities in this system is $n + 2^n - 1 - |\mathcal{B}|$, the above condition yields $|\mathcal{B}| \leqslant n$. □

Theorem 2.7 (Bondareva-Shapley theorem, sharp form). Let (N, v) be a game. Its core is nonempty if and only if for any minimal balanced collection $\mathcal{B}$ with system of balancing weights $(\lambda_A)_{A\in\mathcal{B}}$, we have $v(N) \geqslant \sum_{A\in\mathcal{B}}\lambda_A v(A)$. Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.

Proof. Every $\lambda \in F$ is a convex combination of extreme points $\lambda^1, \ldots, \lambda^k$:

$$\lambda = \alpha_1\lambda^1 + \cdots + \alpha_k\lambda^k.$$

For each λ^i, the inequality $v(N) \geqslant \sum_{A\in\mathcal{B}}\lambda^i_A v(A)$ is valid, therefore

$$\underbrace{\sum_{i=1}^k \alpha_i v(N)}_{v(N)} \geqslant \sum_{A\in\mathcal{B}} v(A) \underbrace{\sum_{i=1}^k \alpha_i\lambda^i_A}_{\lambda_A}.$$

Hence v is balanced, and by Theorem 2.6 its core is nonempty.

The converse statement is obvious.

[this part is optional] It remains to prove that none of the inequalities in the system $\{\sum_{A\in\mathcal{B}}\lambda^{\mathcal{B}}_A v(A) \leqslant v(N), \mathcal{B}$ minimal balanced $, \mathcal{B} \neq \{N\}\}$ is redundant. From Farkas' Lemma II, it suffices to prove that choosing any inequality $\sum_{A\in\mathcal{B}^*}\lambda^{\mathcal{B}^*}_A v(A) \leqslant v(N)$ in the system, a conic combination of the left members of the remaining ones cannot give the left member of the chosen inequality. In symbols, for all nonnegative coefficients $\gamma^{\mathcal{B}}$, with $\mathcal{B}$ minimal balanced and different from $\mathcal{B}^*$, the equalities

$$\sum_{\mathcal{B}\neq\mathcal{B}^*, \mathcal{B}\ni S} \gamma^{\mathcal{B}}\lambda^{\mathcal{B}}_S = \begin{cases} \lambda^{\mathcal{B}^*}_S, & S \in \mathcal{B}^* \\ 0, & \text{otherwise} \end{cases}$$

cannot hold simultaneously. Choose $S \in \mathcal{B}^*$. Then there exists some minimal balanced collection $\tilde{\mathcal{B}} \neq \mathcal{B}^*$ such that $\tilde{\mathcal{B}} \ni S$ and $\gamma^{\tilde{\mathcal{B}}} > 0$ (otherwise $0 < \lambda^{\mathcal{B}^*}_S = \sum_{\mathcal{B}\neq\mathcal{B}^*,\mathcal{B}\ni S}\gamma^{\mathcal{B}}\lambda^{\mathcal{B}}_S$ is not possible). Because $\tilde{\mathcal{B}} \neq \mathcal{B}^*$ and $\tilde{\mathcal{B}} \subset \mathcal{B}^*$ is impossible by minimality, there exists $T \in \tilde{\mathcal{B}}, T \notin \mathcal{B}^*$. Therefore

$$0 = \sum_{\mathcal{B}\neq\mathcal{B}^*, \mathcal{B}\ni T} \gamma^{\mathcal{B}}\lambda^{\mathcal{B}}_T \geqslant \gamma^{\tilde{\mathcal{B}}}\lambda^{\tilde{\mathcal{B}}}_T > 0,$$

a contradiction. □

Example 2.5. We enumerate the minimal balanced collections for $N = \{1, 2, 3, 4\}$. Every partition is obviously minimal, and there are 15 partitions of N. Apart these, the following are minimal balanced collections:

$$\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}, \qquad \text{with } \lambda = \Big(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\Big)$$
$$\mathcal{B} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}, \qquad \text{with } \lambda = \Big(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\Big)$$
$$\mathcal{B} = \{\{1,2\},\{1,3\},\{2,3\},\{4\}\}, \qquad \text{with } \lambda = \Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\Big)$$
$$\mathcal{B} = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}, \qquad \text{with } \lambda = \Big(\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)$$
$$\mathcal{B} = \{\{1,2,3\},\{1\},\{3,4\},\{2,4\}\} \qquad \text{with } \lambda = \Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)$$

and those obtained by permutations.

2.5 The Weber set

Let $\sigma : N \to N$ be a permutation on N (order in which the players enter the game). We denote by $\mathfrak{S}(N)$ the set of permutations on N. There is a one-to-one correspondence between each permutation σ and each *maximal chain* in $(2^N, \subseteq)$ (i.e., a sequence of n subsets $\varnothing = S_0 \subset S_1 \subset \cdots \subset S_n = N$), defined by

$$\begin{aligned} S_1 &= \{\sigma(1)\} \\ S_2 \setminus S_1 &= \{\sigma(2)\} \\ \vdots &= \vdots \\ S_n \setminus S_{n-1} &= \{\sigma(n)\}, \end{aligned}$$

that is, $S_i = \{\sigma(1), \ldots, \sigma(i)\}$. Next we associate to σ and v its *marginal vector* $m^{\sigma,v} \in \mathbb{R}^N$ (also denoted m^σ if there is no fear of ambiguity) defined by

$$m^{\sigma,v}_{\sigma(i)} = v(S_i) - v(S_{i-1}) \qquad (i \in N). \tag{2.2}$$

It is easy to check that this is equivalent to

$$\sum_{j=1}^{i} m^{\sigma,v}_{\sigma(j)} = m^{\sigma,v}(S_i) = v(S_i) \qquad (i \in N). \tag{2.3}$$

Example 2.6. Let $N = \{1,2,3\}$ and the permutation σ defined by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. Then

$$S_1 = \{2\}, \quad S_2 = \{2,3\}, \quad S_3 = \{1,2,3\}.$$

The marginal vector is

$$\begin{aligned} m^\sigma_1 &= v(\{1,2,3\}) - v(\{2,3\}) \\ m^\sigma_2 &= v(\{2\}) \\ m^\sigma_3 &= v(\{2,3\}) - v(\{2\}). \end{aligned}$$

Definition 2.11. The *Weber set* of a game (N, v) is the convex hull of its marginal vectors:

$$W(v) = \mathrm{conv}\{m^\sigma \mid \sigma \in \mathfrak{S}(N)\}.$$

Example 2.7. Let $N = \{1, 2, 3\}$ and consider the game v defined by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	1	1	-1	3

We obtain

σ	m_1^σ	m_2^σ	m_3^σ
1,2,3	0	1	2
1,3,2	0	2	1
2,1,3	1	0	2
2,3,1	4	0	-1
3,1,2	1	2	0
3,2,1	4	-1	0

The core is given by the set of $x \in \mathbb{R}^3$ such that

$$\begin{aligned} x_1 + x_2 \geqslant 1 &\quad\Leftrightarrow\quad x_3 \leqslant 2 \\ x_1 + x_3 \geqslant 1 &\quad\Leftrightarrow\quad x_2 \leqslant 2 \\ x_2 + x_3 \geqslant -1 &\quad\quad\quad\;\; \text{(always true)} \end{aligned}$$

The Weber set together with the core are represented on Figure 2.1. We can see that $C(v) \subseteq W(v)$.

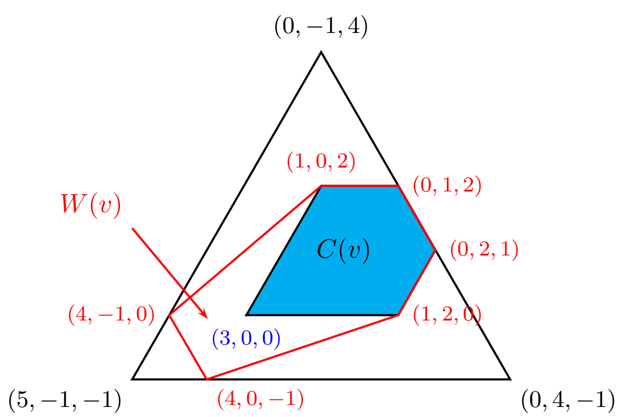


Figure 2.1: The core and the Weber set

Theorem 2.8. For any game (N, v), $C(N, v) \subseteq W(N, v)$.

The proof is based on the separating hyperplane theorem: let $Z \subseteq \mathbb{R}^n$ be a closed convex set, and let $x \in \mathbb{R}^n \setminus Z$. Then there exists $y \in \mathbb{R}^n$ such that $\langle y, z \rangle > \langle y, x \rangle$ for all $z \in Z$, where $\langle \cdot \rangle$ denotes a scalar product.

Proof. Suppose there exists $x \in C(v) \setminus W(v)$. By the separating hyperplane Theorem, there exists $y \in \mathbb{R}^n$ such that

$$\langle w, y\rangle > \langle x, y\rangle \qquad (w \in W(v)).$$

Let $\pi \in \mathfrak{S}(N)$ be a permutation such that $y_{\pi(1)} \geqslant y_{\pi(2)} \geqslant \cdots \geqslant y_{\pi(n)}$. In particular for $w = m^\pi$, we find

$$\langle m^\pi, y\rangle > \langle x, y\rangle. \tag{2.4}$$

Since $x \in C(v)$, we have

$$\begin{aligned}
\langle m^\pi, y\rangle &= \sum_{i=1}^{n} y_{\pi(i)}\big(v(\{\pi(1),\dots,\pi(i)\}) - v(\{\pi(1),\dots,\pi(i-1)\})\big)\\
&= y_{\pi(n)}v(N) - y_{\pi(1)}v(\varnothing) + \sum_{i=1}^{n-1}(y_{\pi(i)} - y_{\pi(i+1)})v(\{\pi(1),\dots,\pi(i)\})\\
&\leqslant y_{\pi(n)}x(N) + \sum_{i=1}^{n-1}(y_{\pi(i)} - y_{\pi(i+1)})x(\{\pi(1),\dots,\pi(i)\})\\
&= \sum_{i=1}^{n} y_{\pi(i)}x(\{\pi(1),\dots,\pi(i)\}) - \sum_{i=2}^{n} y_{\pi(i)}x(\{\pi(1),\dots,\pi(i-1)\})\\
&= \sum_{i=1}^{n} y_{\pi(i)}x_{\pi(i)} = \langle y, x\rangle,
\end{aligned}$$

which contradicts (2.4). □

Theorem 2.9. Let (N, v) be a game. The following propositions are equivalent.

(i) v is convex;

(ii) $m^\sigma \in C(v)$ for all $\sigma \in \mathfrak{S}(N)$;

(iii) $C(v) = W(v)$;

(iv) $\mathrm{ext}(C(v)) = \{m^\sigma \mid \sigma \in \mathfrak{S}(N)\}$.

Proof. (i)⇒(ii) Let us take $\sigma = Id$ for ease of notation. Let $S \subseteq N$ defined by $S = \{i_1, \dots, i_s\}$ (put in numerical order). Since v is convex, we obtain:

$$v(\{i_1,\dots,i_k\}) - v(\{i_1,\dots,i_{k-1}) \leqslant v(\{1,2,\dots,i_{k-1},i_k\} - v(\{1,2,\dots,i_{k-1}\}) = m^\sigma_{i_k}.$$

If we add all these inequalities for $k = 1$ to s, we obtain

$$v(S) = v(\{i_1,\dots,i_s\}) \leqslant \sum_{k=1}^{s} m^\sigma_{i_k} = \sum_{i\in S} m^\sigma_i,$$

which proves that $m^\sigma \in C(v)$.

(ii)⇒(i) Let $S, T \subseteq N$. Let us order the players of N as follows:

$$N = \{\underbrace{i_1,\dots,i_k}_{S\cap T}, \underbrace{i_{k+1},\dots,i_\ell}_{T\setminus S}, \underbrace{i_{\ell+1},\dots,i_s}_{S\setminus T}, \underbrace{i_{s+1},\dots,i_n}_{N\setminus(S\cup T)}\},$$

which defines a permutation σ. Since $m^\sigma \in C(v)$, we have:

$$\begin{aligned}
v(S) \leqslant \sum_{i\in S} m_i^\sigma &= \sum_{j=1}^{k} m_{i_j}^\sigma + \sum_{j=\ell+1}^{s} m_{i_j}^\sigma \\
&= v(\{i_1,\dots,i_k\}) + \big[v(\{i_1,\dots,i_{\ell+1}\}) - v(\{i_1,\dots,i_\ell\})\big] \\
&\quad + \cdots + \big[v(\{i_1,\dots,i_s\}) - v(\{i_1,\dots,i_{s-1}\})\big] \\
&= v(\{i_1,\dots,i_k\}) - v(\{i_1,\dots,i_\ell\}) + v(\{i_1,\dots,i_s\}) \\
&= v(S\cap T) - v(T) + v(S\cup T).
\end{aligned}$$

Hence, v is convex.

(iv)⇒(ii) Clear.

(ii)⇒(iii) $C(v) \subseteq W(v)$ is already shown. (ii) implies the converse inclusion.

(iii)⇒(iv) (iii) shows that every extreme point is a marginal vector. It remains to prove that every marginal vector is an extreme point.

This comes from the fact that m^σ satisfies at least n equalities in the system $m^\sigma(S) \geqslant v(S)$, corresponding to those S in the maximal chain induced by σ. This system being triangular, it has a unique solution, and therefore it defines an extreme point. □

Chapter 3

The Shapley value

3.1 Definition

Definition 3.1. Let (N, v) be a game. The *Shapley value* of the game is a vector in $\mathbb{R}^N$ given by

$$\Phi(N, v) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}(N)} m^\sigma \tag{3.1}$$

where m^σ denotes the marginal vector of v for permutation σ.

Some remarks:

(i) The Shapley value is a point-type solution. It always exists. Generally speaking, a *value* is a mapping $\Phi : \mathcal{G}(N) \to \mathbb{R}^N$, where $\mathcal{G}(N)$ denotes the set of all games on N.

(ii) Interpretation: the Shapley value is the average of the marginal contributions (or expected value w.r.t. a uniform distribution) over all possible orders of the players.

(iii) With $n = 2$, the expression becomes:

$$\begin{aligned}
\Phi_1(v) &= \frac{1}{2}\Big(v(\{1\}) + v(\{1,2\}) - v(\{2\})\Big) \\
&= v(\{1\}) + \frac{v(N) - v(\{1\}) - v(\{2\})}{2} \\
\Phi_2(v) &= \frac{1}{2}\Big(v(\{2\}) + v(\{1,2\}) - v(\{1\})\Big) \\
&= \underbrace{v(\{2\})}_{\text{what has each player}} + \underbrace{\frac{v(N) - v(\{1\}) - v(\{2\})}{2}}_{\text{equal share of the gain/loss of cooperation}}
\end{aligned}$$

(iv) $\Phi_i(v) = v(\{i\})$ if v is additive.

Example 3.1. We compute the Shapley value for the 3 cities example.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	90	100	120	220

We have:

order σ	m_1^σ	m_2^σ	m_3^σ
1,2,3	0	90	130
1,3,2	0	120	100
2,1,3	90	0	130
2,3,1	100	0	120
3,1,2	100	120	0
3,2,1	100	120	0
$\Phi(v)$	65	75	80

Let us now compute the Shapley value for the glove game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	1	1	1

We obtain:

order σ	m_1^σ	m_2^σ	m_3^σ
1,2,3	0	0	1
1,3,2	0	0	1
2,1,3	0	0	1
2,3,1	0	0	1
3,1,2	1	0	0
3,2,1	0	1	0
$\Phi(v)$	1/6	1/6	2/3

Note that in the glove game, the core is the point $\{(0,0,1)\}$. Hence, in general the Shapley value does not lie in the core. Moreover, one can find examples where the Shapley value is not individually rational (try!). The following result is easy to obtain.

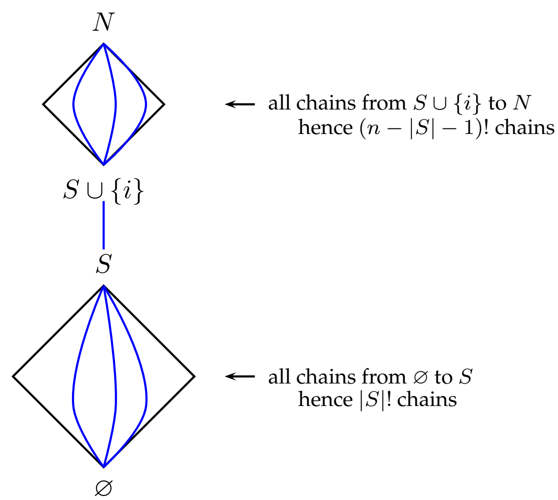
Theorem 3.1. If (N,v) is convex, then the Shapley value is the (weighted) barycenter of the core, and therefore lies in it.

Proof. If v is convex, then

$$C(v) = \mathrm{conv}\{m^\sigma \mid \sigma \in \mathfrak{S}(N)\}$$

and $\Phi(v) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}(N)} m^\sigma$. □

It is possible to considerably simplify the computation of $\Phi(v)$, by remarking that for $\Phi_i(v)$, the sum in (3.1) is composed only by terms of the form $v(S\cup\{i\}) - v(S)$, and that a given term is repeated a certain number of times (see figure below).



Putting $|S| = s$, we therefore have $(n-s-1)!s!$ chains which contain $v(S \cup \{i\}) - v(S)$. Consequently,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!}[v(S \cup \{i\}) - v(S)],$$

which is the usual formula.

Observe that

$$\frac{(n-s-1)!s!}{n!} = \frac{1}{n}\frac{1}{\binom{n-1}{s}}$$

(recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$), which suggests the following probabilistic interpretation: pick at random a coalition $S \subseteq N \setminus \{i\}$ in the following way:

(i) choose the size s of S according to a uniform distribution on $0, 1, \ldots, n-1$.

(ii) then, among the sets of size s, choose one at random in a uniform way.

3.2 Characterization

We introduce the following properties for a value Ψ on $\mathcal{G}(N)$.

- **Efficiency (E)**: $\sum_{i \in N} \Psi_i(v) = v(N)$.
- We say that a player i is *null* if $v(S \cup \{i\}) = v(S)$ for every coalition $S \subseteq N \setminus \{i\}$. In particular, $v(\{i\}) = 0$ for a null player.

  Null player property (N): $\Psi_i(v) = 0$ if i is a null player.
- A player i is *dummy* if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every coalition $S \subseteq N \setminus \{i\}$.

  Dummy player property (D): $\Psi_i(v) = v(\{i\})$ if i is a dummy player.

Observe that if Ψ satisfies (D) then it satisfies (N) (why?).

- Two players i, j are *symmetric* if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$.

  Symmetry (S): $\Psi_i(v) = \Psi_j(v)$ if i, j are symmetric.

- Let $\sigma \in \mathfrak{S}(N)$ be a permutation on N. We define the game v^σ on N by

$$v^\sigma(\sigma(S)) = v(S) \qquad (S \subseteq N)$$

  (i.e., $v^\sigma(S) = v(\sigma^{-1}(S))$.

  Anonymity (A): for all $\sigma \in \mathfrak{S}(N)$,

$$\Psi_{\sigma(k)}(v^\sigma) = \Psi_k(v) \qquad (k \in N).$$

  Observe that if Ψ satisfies (A) then it satisfies (S). (A) means that the numbering of the players has no impact on the value for the players.

- **Additivity (ADD)**: $\Psi(v + w) = \Psi(v) + \Psi(w)$ for all $v, w \in \mathcal{G}(N)$.

  Interpretation: (N, v) is played today and (N, w) is played tomorrow. The player i obtains in total $\Psi_i(v) + \Psi_i(w)$. One can consider that the game $v + w$ has been played.

- **Linearity (L)**: $\Psi(v + \alpha w) = \Psi(v) + \alpha\Psi(w)$ for every $v, w \in \mathcal{G}(N)$ and $\alpha \in \mathbb{R}$.

Exercise 3.1. Prove that the Shapley value satisfies all of the above properties.

Theorem 3.2. (Shapley, 1953) Let $\Psi : \mathcal{G}(N) \to \mathbb{R}^N$. Then Ψ is the Shapley value iff it satisfies (E), (N), (S) and (ADD).

Proof. $\Rightarrow$) see Exercise 3.1.
$\Leftarrow$) We use the decomposition of game on the basis of unanimity games:

$$v = \sum_{\substack{T \subseteq N \\ T \neq \varnothing}} m_T u_T$$

By (ADD), $\Psi(v) = \sum_{\substack{T \subseteq N \\ T \neq \varnothing}} \Psi(m_T u_T)$. It is enough to show that $\Psi(\alpha u_T) = \Phi(\alpha u_T)$ for every $\alpha \in \mathbb{R}$, $\varnothing \neq T \subseteq N$. Let $\alpha \in \mathbb{R}$, $\varnothing \neq T \subseteq N$ to be fixed. Remark that $i \in N \setminus T$ is a null player for the game αu_T:

$$\alpha u_T(S \cup \{i\}) = \alpha u_T(S) \qquad (S \subseteq N \setminus \{i\}).$$

Consequently, by (N)

$$\Psi_i(\alpha u_T) = 0 = \Phi_i(\alpha u_T) \qquad (i \in N \setminus T). \tag{3.2}$$

Let us take distinct $i, j \in T$, supposing $|T| \geqslant 2$. Then i, j are symmetric in the game u_T:

$$\alpha u_T(S \cup \{i\}) = \alpha u_T(S \cup \{j\}) \qquad (S \subseteq N \setminus \{i, j\})$$

Therefore, $\Psi_i(\alpha u_T) = \Psi_j(\alpha u_T)$ and $\Phi_i(\alpha u_T) = \Phi_j(\alpha u_T)$. By (E) and (3.2), we deduce that

$$\Psi_i(\alpha u_T) = \Phi_i(\alpha u_T) = \frac{\alpha}{|T|} \qquad (i \in T). \tag{3.3}$$

Lastly, supposing $|T| = 1$, say $T = \{i\}$, by (E) we obtain

$$\Psi_i(\alpha u_T) = \Phi_i(\alpha u_T) = \alpha.$$

Hence (3.3) is valid also for $|T| = 1$. Combining (3.2) and (3.3) we deduce that $\Phi(\alpha u_T) = \Psi(\alpha u_T)$. □

3.3 Characterization without additivity

Additivity is a usual property in mathematics but not very natural in this context of cooperative game theory. Young has proposed a characterization without (ADD). It is founded on the following property:

Strong Monotonicity (SM): Let v, w be two games on N satisfying

$$v(S \cup \{i\}) - v(S) \geqslant w(S \cup \{i\}) - w(S) \qquad (S \subseteq N \setminus \{i\}).$$

Then $\Psi_i(v) \geqslant \Psi_i(w)$.

Obviously, the Shapley value satisfies (SM).

Theorem 3.3. Let $\Psi : \mathcal{G}(N) \to \mathbb{R}^N$. Then Ψ is the Shapley value iff it satisfies (E), (S) and (SM).

Proof. Let $v \in \mathcal{G}(N)$. We define

$$\mathcal{D}(v) = \{S \subseteq N \mid \exists T \subseteq S, v(T) \neq 0\}.$$

We will show that $\Psi(v) = \Phi(v)$ by induction on $|\mathcal{D}(v)|$.

1. If $|\mathcal{D}(v)| = 0$ then $v = 0$ and therefore by (E) and (S) $\Psi(v) = 0 = \Phi(v)$.

2. Suppose the property to be true till $|\mathcal{D}(v)| = k$ and consider v such that $|\mathcal{D}(v)| = k + 1$. Let us call $\mathcal{D}^m(v)$ the set of minimal elements of $\mathcal{D}(v)$:

$$\mathcal{D}^m(v) = \{S \subseteq N \mid v(S) \neq 0 \text{ and } T \subset S \Rightarrow v(T) = 0\}.$$

Let $S \in \mathcal{D}^m(v)$, define v_S on N by $v_S(T) = v(S \cap T)$, $T \subseteq N$, and put $w = v - v_S$. Then $|D(w)| \leq k$ and for all $i \in N \setminus S$,

$$w(T \cup i) - w(T) = v(T \cup i) - v(T) \qquad (T \subseteq N \setminus i)$$

Indeed, w and v only differ on supersets of S, and if T is a superset of S, so is $T \cup i$, while if T is not a superset of S, neither $T \cup i$ is a superset of S.

Hence, by (SM), we have $\Psi_i(w) = \Psi_i(v)$, and by induction hypothesis, $\Psi_i(w) = \Phi_i(w)$ for all $i \notin S$. By (SM) applied to Φ, we also have $\Phi_i(w) = \Phi_i(v)$, and therefore $\Psi_i(v) = \Phi_i(v)$ for all $i \in N \setminus S$.

In summary we have shown that $\Psi_i(v) = \Phi_i(v)$ for all $i \in N \setminus S_0$, with

$$S_0 = \bigcap \{S \mid S \in \mathcal{D}^m(v)\}.$$

We have $v(T) = 0$ if $S_0 \setminus T \neq \varnothing$ (indeed, $T \not\supseteq S_0$ implies $T \not\supseteq S$, for all $S \in \mathcal{D}^m(v)$, which implies $T \notin \mathcal{D}(v)$). Hence, if $i, j \in S_0$, they are symmetric (since $v(T \cup i) = v(T \cup j) = 0$), hence by (S) we obtain $\Psi_i(v) = \Psi_j(v)$ and $\Phi_i(v) = \Phi_j(v)$ for all $i, j \in S_0$. Consequently, by (E) we obtain $\Psi_i(v) = \Phi_i(v)$ for all $i \in S_0$. □

3.4 Potential

We denote by $\mathcal{G} = \bigcup_{\substack{N \subseteq \mathbb{N} \\ |N| < \infty}} \mathcal{G}(N)$ the set of all games with a finite number of players.

Definition 3.2. (Hart and Mas-Colell, 1989) A *potential* is a function $P : \mathcal{G} \to \mathbb{R}$ satisfying:

(i) $P(\varnothing, v) = 0$

(ii) $\sum_{i \in N} D^i P(N, v) = v(N)$, for all $(N, v) \in \mathcal{G}$,

with $D^i P(N, v) = P(N, v) - P(N \setminus i, v)$, where v in the last term is with some abuse of notation the restriction of v to $N \setminus i$.

The *gradient* of P is the vector $(D^i P(N, v))_{i \in N}$. It is an efficient vector.

Theorem 3.4. There exists a unique potential $P : \mathcal{G} \to \mathbb{R}$. $P(N, v)$ is determined by the subgames (S, v), $S \subseteq N$, by applying (ii) recursively.

Proof. We have for $|N| = 1$, $P(\{i\}, v) = v(\{i\})$. For $|N| = 2$, we obtain:

$$P(\{i, j\}, v) = \frac{1}{2}\Big(v(\{i, j\}) + v(\{i\}) + v(\{j\})\Big)$$

Finally:

$$P(N, v) = \frac{v(N) + \sum_{i \in N} P(N \setminus i, v)}{|N|}$$

□

Theorem 3.5. For every game $(N, v) \in \mathcal{G}$, for all $i \in N$, $D^i P(N, v) = \Phi_i(N, v)$ (Shapley value).

Proof. Let $(N, v) \in \mathcal{G}$ and express it in the basis of unanimity games:

$$v = \sum_{\varnothing \neq T \subseteq N} \alpha_T u_T.$$

Let us define a function $P^* : \mathcal{G} \to \mathbb{R}$ by

$$P^*(N, v) = \sum_{\varnothing \neq T \subseteq N} \frac{\alpha_T}{|T|}$$

and $P^*(\varnothing, v) = 0$. Take $i \in N$. If $|N| = 1$, $P^*(N, v) = D^i P^*(N, v) = v(N)$. If $|N| \geqslant 2$, then

$$P^*(N \setminus i, v) = \sum_{\varnothing \neq T \subseteq N \setminus i} \frac{\alpha_T}{|T|}$$

therefore

$$\sum_{i \in N} D^i P^*(N, v) = \sum_{i \in N} \sum_{T \ni i} \frac{\alpha_T}{|T|} = \sum_{\varnothing \neq T \subseteq N} \sum_{i \in T} \frac{\alpha_T}{|T|} = \sum_{\varnothing \neq T \subseteq N} \alpha_T = v(N).$$

Hence P^* is a potential, and by uniqueness, $P^* = P$. Furthermore, we recognize in $D^i P^*(N, v) = \sum_{T \ni i} \frac{\alpha_T}{|T|}$ the Shapley value. □

Proposition 3.1. For any game $(N, v) \in \mathcal{G}$,

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S).$$

(proof is left as an exercise)

3.5 Reduced games

The basic idea is the following: In a game (N, v), consider a coalition $S \neq N$ and the game "induced" by S (called reduced game). If the players apply the same solution concept to N or S, their benefit should not change. This property is called "consistency".

Definition 3.3. Let Ψ be a value defined on $\mathcal{G}$ and let $(N, v) \in \mathcal{G}$. For every $U \subseteq N$, $U \neq \varnothing$, the *reduced game* $(N \setminus U, v_{U,\Psi})$ is defined by

$$v_{U,\Psi}(S) = \begin{cases} v(S \cup U) - \sum_{\in U} \Psi_k(S \cup U, v), & S \subseteq N \setminus U, S \neq \varnothing \\ 0, & S = \varnothing. \end{cases}$$

Definition 3.4. A value Ψ is *consistent* if for every game (N, v), every $U \subseteq N$, $U \neq \varnothing$,

$$\Psi_i(N \setminus U, v_{U,\psi}) = \Psi_i(N, v) \qquad (i \in N \setminus U).$$

Lemma 3.1. Let $(N, v) \in \mathcal{G}$. Suppose that $Q : 2^N \to \mathbb{R}$ satisfies

$$\sum_{i \in S} (Q(S) - Q(S \setminus i)) = v(S) \qquad (S \subseteq N, S \neq \varnothing)$$

Then for all $S \subseteq N$,

$$Q(S) = P(S, v) + Q(\varnothing). \tag{3.4}$$

Proof. By induction on $|S|$. For $|S| = 0$, the property is clearly true. Let T, $|T| > 0$, and suppose that (3.4) is true for all S, $|S| < |T|$. We obtain:

$$\begin{aligned} Q(T) =& \frac{1}{|T|}\Big(v(T) + \sum_{i \in T} Q(T \setminus i)\Big) \\ &= \frac{1}{|T|}\Big(v(T) + |T|Q(\varnothing) + \sum_{i \in T} P(T \setminus i, v)\Big) \\ & Q(\varnothing) + \underbrace{\frac{1}{|T|}\Big(v(T) + \sum_{i \in T} P(T \setminus i, v)\Big)}_{P(T,v)}. \end{aligned}$$

$\square$

Lemma 3.2. The Shapley value is consistent.

Proof. Let $(N, v) \in \mathcal{G}$, $U \subseteq N$, $U \neq \varnothing$. We have to show that

$$\Phi_i(N \setminus U, v_{U,\Phi}) = \Phi_i(N, v) \qquad (i \in N \setminus U).$$

We have

$$\begin{aligned} v_{U,\Phi}(S) &= v(S \cup U) - \sum_{i \in U} \Phi_i(S \cup U, v) = \sum_{i \in S} \Phi_i(S \cup U), v) \\ &= \sum_{i \in S} \underbrace{[P(S \cup U, v) - P((S \cup U) \setminus i, v)]}_{D^i P(S \cup U, v)} \end{aligned}$$

Let us define $Q(S) = P(S \cup U, v)$ for all $S \subseteq N \setminus U$. By the preceding lemma applied to $(N \setminus U, v_{U,\Phi})$, we obtain:

$$\underbrace{Q(S)}_{P(S \cup U, v)} = P(S, v_{U,\Phi}) + \underbrace{Q(\varnothing)}_{P(U,v)} \qquad (S \subseteq N \setminus U)$$

Then

$$\begin{aligned} \Phi_i(N \setminus U, v_{U,\Phi}) &= P(N \setminus U, v_{U,\Phi}) - P((N \setminus U) \setminus i, v_{U,\Phi}) \\ &= P(N, v) - P(N \setminus i, v) = \Phi_i(N, v). \end{aligned}$$

□

Definition 3.5. Ψ is *standard for 2-players games* if for all $(\{i,j\}, v)$,

$$\begin{aligned} \Psi_i(\{i,j\}, v) &= \frac{1}{2}\big(v(\{i,j\}) + v(\{i\}) - v(\{j\})\big) \\ \Psi_j(\{i,j\}, v) &= \frac{1}{2}\big(v(\{i,j\}) + v(\{j\}) - v(\{i\})\big) \end{aligned}$$

Theorem 3.6. Let Ψ be a value on $\mathcal{G}$. Then Ψ is the Shapley value if and only if Ψ is consistent and standard for 2-players games.

3.6 The Banzhaf value

There are other values than the Shapley value. One of the best known among these is the Banzhaf value. It is often used in voting games, under the name of Banzhaf (power) index.

In voting games (i.e., simple games), a value is usually called a *power index*. For example, the Shapley value is called the *Shapley-Shubik index*. A central notion in voting is the notion of swing. A *swing* for player i is a coalition S such that $i \in S$, S is winning and $S \setminus \{i\}$ is losing. In other words, i is a pivot player or key player in S. We denote by θ_i the number of swings for player i. The *Banzahf index* (or *normalized Banzhaf-Coleman* index) is defined as follows:

$$\beta_i(N, v) = \frac{\theta_i}{\sum_{j \in N} \theta_j} \qquad (i \in N).$$

Observe that $\theta_i = \sum_{S \subseteq N \setminus i}(v(S \cup i) - v(S))$ for a simple game. This yields the extension of the Banzhaf index to any TU-game, called the *Banzhaf value*:

$$\Psi_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} (v(S \cup i) - v(S)) \qquad (i \in N).$$

It is easy to check that the Banzahf value satisfies (N), (D), (S), (A), (SM), (ADD), (L), which implies that (E) cannot be satisfied (as this would characterize the Shapley value). While there is no inconvenience that a power index does not satisfy (E), as a value is a sharing of the total benefit $v(N)$, it is rather inconvenient that (E) is not satisfied by the Banzhaf value. This is why the *normalized Banzhaf value* has been introduced:

$$\tilde{\Psi}_i(N, v) = \frac{\Psi_i(v)}{\sum_{j \in N} \Psi_j(v)} \qquad (i \in N).$$

The normalized Banzhaf value is efficient, but loses the additivity (and hence linearity) property.

Chapter 4

The nucleolus

Solution concepts seen so far are not completely satisfactory:

- The core can be very large, or empty.
- The Shapley value is not always in the core.

The nucleolus (Schmeidler, 1969) is unique and always exists. Moreover, it always belongs to the core when it is nonempty.

4.1 Definition

Let (N, v) be a game, and $X \subseteq \mathbb{R}^N$ a set of payment vectors. For all $x \in X$, $S \subseteq N$, $S \neq \varnothing$, we define the *excess* of S at x by:

$$e(S, x) = v(S) - x(S)$$

(regret, dissatisfaction of coalition S w.r.t. payment x). We define the *vector* $\theta(x)$ *of ordered excesses* at x:

$$\theta(x) = (e(S_1, x), \ldots, e(S_{2^n-1}, x))$$

ordered such that $e(S_1, x) \geqslant e(S_2, x) \geqslant \cdots \geqslant e(S_{2^n-1}, x)$. The *nucleolus* of v w.r.t. X is the set

$$\mathcal{N}(N, v, X) = \{x \in X \mid \theta(y) \succcurlyeq_{\text{lex}} \theta(x)\ \forall y \in X\},$$

with $\succcurlyeq_{\text{lex}}$ the lexicographic order.

Interpretation: we look for payment vectors which minimize the maximum excess.

Definition 4.1. (i) The *nucleolus* of v is the nucleolus w.r.t. $X = I(N, v)$, the set of imputations. Notation: $\nu(N, v)$.

(ii) The *prenucleolus* of v is the nucleolus w.r.t. $X = PI(N, v)$, the set of preimputations. Notation: $\nu^*(N, v)$.

Some remarks:

(i) The nucleolus may not exist since $I(N, v)$ can be empty, but the prenucleolus always exists.

(ii) If $C(N, v) \neq \varnothing$, every $x \in C(N, v)$ yields $\theta(x) \leqslant 0$, and every x outside the core yields $\theta(x)$ with at least one positive component. Hence the search for a minimal $\theta(x)$ can be restricted to elements of the core. Therefore, the nucleolus and the prenucleolus coincide.

(iii) For the same reason, if $C(N, v) \neq \varnothing$, the nucleolus lies in the core.

4.2 Example

Consider the following game:

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	4	4	4	8	12	16	24

The core is nonempty since $(8, 8, 8) \in C(v)$. This gives for the excesses:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$e(S,(8,8,8))$	-4	-4	-4	-8	-4	0

The maximum excess is 0. it can be easily diminished by giving more to players 2 and 3, to the detriment of player 1. For example:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$e(S,(6,9,9))$	-2	-5	-5	-7	-3	-2

The maximum excess is -2, and it cannot be further diminished: increasing x_2 or x_3 obliges to diminish x_1, which would have as effect to increase $e(\{1\}, x)$.

Let us remark that $\{\{1\}, \{2,3\}\}$ (red) is a balanced collection.

On the other hand, it is possible to minimize the second greatest excess -3, by giving more to players 1,3 to the detriment of player 2. However, $x_1 = 6$ cannot be further increased.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$e(S,(6,8,10))$	-2	-4	-6	-6	-4	-2

It is not possible to further diminish the second excess -4 (because we should diminish x_2 and therefore $e(\{2\}, x) > -4$), nor the third one -6 (idem with x_3).

Finally, $(6, 8, 10)$ is the nucleolus and the prenucleolus of the game. Remark that the collections $\{\{1\}, \{2,3\}, \{2\}, \{1,3\}\}$ (red and blue) and $\{\{1\}, \{2,3\}, \{2\}, \{1,3\}, \{3\}, \{1,2\}\}$ (red, blue, green) are balanced.

Generally speaking, the computation of the nucleolus is not simple.

4.3 Existence and uniqueness

Theorem 4.1. Let $X \subseteq \mathbb{R}^N$, compact and nonempty. Then $\mathcal{N}(N, v, X) \neq \varnothing$.

Proof. The mapping $e(S,\cdot)$ is continuous, and therefore $\theta(\cdot)$ also. Let us define $X_0 := X$, and

$$X_t := \{x \in X_{t-1} \mid \theta_t(y) \geqslant \theta_t(x),\ \forall y \in X_{t-1}\} \quad (t = 1, 2, \ldots, 2^n - 1).$$

By the Weierstraß theorem[1], X_t is nonempty, and also compact[2], for every t. As $\mathcal{N}(N, v, X) = X_{2^n-1}$, the result is shown. □

Theorem 4.2. Let $X \subseteq \mathbb{R}^N$, compact, convex and nonempty. Then $\mathcal{N}(N, v, X)$ is a singleton for all games (N, v).

The proof is based on the following lemma:

Lemma 4.1. Let X be convex, $x, y \in X$, $0 \leqslant \alpha \leqslant 1$. Then

$$\alpha\theta(x) + (1-\alpha)\theta(y) \succcurlyeq_{\text{lex}} \theta(\alpha x + (1-\alpha)y).$$

Proof. (of the Lemma) We order the sets $S_1, \ldots, S_{2^n-1}$ in such a way that

$$\theta(\alpha x + (1-\alpha)y) = (e(S_1, \alpha x + (1-\alpha)y), \ldots, e(S_{2^n-1}, \alpha x + (1-\alpha)y))$$

The right hand-side is equal to $\alpha a + (1-\alpha)b$, with

$$a = (e(S_1, x), \ldots, e(S_{2^n-1}, x)), \quad b = (e(S_1, y), \ldots, e(S_{2^n-1}, y))$$

Since $\theta(x) \succcurlyeq_{\text{lex}} a$ and $\theta(y) \succcurlyeq_{\text{lex}} b$, we obtain:

$$\alpha\theta(x) + (1-\alpha)\theta(y) \succcurlyeq_{\text{lex}} \alpha a + (1-\alpha)b = \theta(\alpha x + (1-\alpha)y)$$

□

Proof. (of the Theorem) By Theorem 4.1, the nucleolus is nonempty. Let $x, y \in \mathcal{N}(N, v, X)$ and $0 < \alpha < 1$. Then $\theta(x) = \theta(y)$ and by the Lemma

$$\theta(\alpha x + (1-\alpha)y) \preccurlyeq_{\text{lex}} \alpha\theta(x) + (1-\alpha)\theta(y) = \theta(x).$$

As x realizes the minimum of the $\theta(\cdot)$'s, it follows that

$$\theta(\alpha x + (1-\alpha)y) = \theta(x) = \theta(y).$$

With the notation of the Lemma, we have then

$$\theta(x) = \theta(y) = \alpha a + (1-\alpha)b.$$

As $\theta(x) \succcurlyeq_{\text{lex}} a$, $\theta(y) \succcurlyeq_{\text{lex}} b$, we obtain $a = \theta(x)$, $b = \theta(y)$. Since a and b are ordered in the same way, so are $\theta(x)$ and $\theta(y)$, and finally $x = y$. □

As a consequence of Theorem 4.2, the nucleolus of any game (N, v) with a nonempty set of imputations exists and is unique, since $I(N, v)$ is compact and convex. However, the theorem is of no help for proving the existence of the prenucleolus since $PI(v)$ is not compact. The result nevertheless holds.

[1] Weierstraß's theorem: A real-valued continuous function on a compact set attains its maximum and minimum. In $\mathbb{R}^n$, compact sets are the closed and bounded sets.

[2] It is bounded as a subset of a bounded set, and it is closed as the inverse image of a closed set by a continuous function.

Proposition 4.1. The prenucleolus exists and is a singleton for any game (N, v).

Proof. Let $y \in PI(v)$ and $\mu := \max_{S \subseteq N} e(S, y)$. Let us define

$$X = \{x \in PI(v) \mid e(S, x) \leqslant \mu, \ \forall S \subseteq N\}.$$

Then X is nonempty, convex and compact. By application of Theorem 4.2, it follows that $\mathcal{N}(N, v, X)$ is a singleton, which coincides with the prenucleolus $\nu^*(N, v)$. □

4.4 The Kohlberg criterion

Nota: all the presentation is done with the prenucleolus. A similar result exists for the nucleolus.

A *side payment* is a vector $y \in \mathbb{R}^N$ such that $y(N) = 0$. For a game (N, v), $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we define

$$\mathcal{D}(\alpha, x, v) = \{S \in 2^N \setminus \{\varnothing\} \mid e(S, x) \geqslant \alpha\}.$$

Theorem 4.3. Let (N, v) be a game and $x \in PI(v)$. The following propositions are equivalent:

(i) $x = \nu^*(N, v)$

(ii) For all α such that $\mathcal{D}(\alpha, x, v) \neq \varnothing$, for all side payment $y \in \mathbb{R}^N$ such that $y(S) \geqslant 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, we have

$$y(S) = 0, \quad \forall S \in \mathcal{D}(\alpha, x, v).$$

Proof. (i)⇒(ii) Let x, α, y satisfy (ii), $x = \nu^*(N, v)$. Let us define $z_\epsilon = x + \epsilon y$ with $\epsilon > 0$. We have $z_\epsilon \in PI(N, v)$. Let us choose $\epsilon^* > 0$ such that for all $S \in \mathcal{D}(\alpha, x, v)$, for all $T \notin \mathcal{D}(\alpha, x, v)$, $T \neq \varnothing$:

$$e(S, z_{\epsilon^*}) > e(T, z_{\epsilon^*}). \tag{4.1}$$

Then, for all $S \in \mathcal{D}(\alpha, x, v)$, we have

$$\begin{aligned} e(S, z_{\epsilon^*}) &= v(S) - (x(S) + \epsilon^* y(S)) \\ &= e(S, x) - \epsilon^* y(S) \leqslant e(S, x). \end{aligned} \tag{4.2}$$

Suppose there exists $S \in \mathcal{D}(\alpha, x, v)$ such that $y(S) > 0$. Then by (4.1) and (4.2), we would have $\theta(x) \succ_{\text{lex}} \theta(z_{\epsilon^*})$, which is impossible.

(ii)⇒(i) Let $x \in P(N, v)$ satisfying (ii), and $z = \nu^*(N, v)$. Let us denote by $\alpha_1 > \cdots > \alpha_p$ the excesses $e(S, x)$ for all $S \in 2^N \setminus \{\varnothing\}$. Put $y = z - x$, i.e., y is a side payment. Since $\theta(x) \succcurlyeq_{\text{lex}} \theta(z)$, we have $e(S, x) = \alpha_1 \geqslant e(S, z)$ for all $S \in \mathcal{D}(\alpha_1, x, v)$, and thus

$$e(S, x) - e(S, z) = (z - x)(S) = y(S) \geqslant 0.$$

By condition (ii), we deduce that $y(S) = 0$ for all $S \in \mathcal{D}(\alpha_1, x, v)$. Suppose now that $y(S) = 0$ for all $S \in \mathcal{D}(\alpha_t, x, v)$ for some $t < p$. Since $\theta(x) \succcurlyeq_{\text{lex}} \theta(z)$, we have

$$e(S, x) = \alpha_{t+1} \geqslant e(S, z) \qquad (S \in \mathcal{D}(\alpha_{t+1}, x, v) \setminus \mathcal{D}(\alpha_t, x, v)).$$

Hence $y(S) \geqslant 0$, and by (ii), $y(S) = 0$ for all $S \in \mathcal{D}(\alpha_{t+1}, x, v)$. It follows that $y(S) = 0$ for all $S \neq \varnothing$, and thus $x = z$. □

Theorem 4.4. (Kohlberg) Let (N, v) be a game and $x \in PI(N, v)$. T.f.a.e.:

(i) $x = \nu^*(N, v)$

(ii) For all α, $\mathcal{D}(\alpha, x, v) \neq \varnothing$ implies that $\mathcal{D}(\alpha, x, v)$ is a balanced collection.

Proof. (ii)⇒(i) Let x satisfy (ii), $\alpha \in \mathbb{R}$ s.t. $\mathcal{D}(\alpha, x, v) \neq \varnothing$, and a side payment y with $y(S) \geqslant 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. As $\mathcal{D}(\alpha, x, v)$ is balanced, there exists $\lambda_S > 0$, $S \in \mathcal{D}(\alpha, x, v)$ such that

$$\sum_{S \in \mathcal{D}(\alpha,x,v)} \lambda_S 1_S = 1_N.$$

Multiplying by y on each side, we obtain

$$\sum_{S \in \mathcal{D}(\alpha,x,v)} \lambda_S y(S) = y(N) = 0$$

which implies that $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. Hence, by Theorem 4.3, we have $x = \nu^*(N, v)$.

(i)⇒(ii) Let $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\alpha, x, v) \neq \varnothing$ for $x = \nu^*(N, v)$. Consider the linear program

$$\begin{aligned}
\text{Max} & \sum_{S \in \mathcal{D}(\alpha,x,v)} y(S) \\
\text{s.t. } & y(S) \geqslant 0, \ \forall S \in \mathcal{D}(\alpha, x, v) \\
& y(N) = 0
\end{aligned}$$

This program has a feasible solution ($y = 0$), and by Theorem 4.3, the optimal value of the objective function is 0. Therefore, by the duality theorem, its dual program has also a feasible solution:

$$\begin{aligned}
& \text{Min } 0 \\
& \text{s.t.} \sum_{S \in \mathcal{D}(\alpha,x,v), S \ni i} -\lambda_S + \lambda_N = \sum_{S \in \mathcal{D}(\alpha,x,v), S \ni i} 1, \ i \in N \\
& \qquad \lambda_S \geqslant 0, \ S \in \mathcal{D}(\alpha, x, v) \\
& \qquad \lambda_N \in \mathbb{R}.
\end{aligned}$$

Hence, for this solution, we have, for every $i \in N$,

$$\lambda_N = \sum_{\substack{S \in \mathcal{D}(\alpha,x,v) \\ S \ni i}} \underbrace{(1 + \lambda_S)}_{>0} > 0.$$

By putting $\lambda'_S = \frac{1+\lambda_S}{\lambda_N}$, we see that the collection $\mathcal{D}(\alpha, x, v)$ is balanced. □

4.5 Computation of the nucleolus

Let us compute the nucleolus $\mathcal{N}(N, v, X)$ for a given $X \subseteq \mathbb{R}^N$ which is a convex polyhedron.

We start by solving the following LP in variables α, x:

$$
\begin{aligned}
&\text{Min } \alpha\\
&\text{s.t. } x(S) + \alpha \geqslant v(S),\ S \subseteq N, S \neq \varnothing\\
&\qquad x \in X.
\end{aligned}
$$

Let α_1 be the optimal value, and $X_1 \subseteq X$ the set of feasible points where the minimum is attained. If $|X_1| = 1$, it is the nucleolus. Otherwise, let $\mathcal{B}_1 = \{S \subseteq N \mid e(S,x) = \alpha_1, x \in X_1\}$. According to the Kohlberg criterion (Theorem 4.4), $\mathcal{B}_1$ is a balanced collection, since for $x \in X_1$, in particular for the nucleolus, $\mathcal{D}(\alpha, x, v) = \mathcal{B}_1$.

Then, we solve the following LP:

$$
\begin{aligned}
&\text{Min } \alpha\\
&\text{s.t. } x(S) + \alpha \geqslant v(S),\ S \in 2^N \setminus \mathcal{B}_1, S \neq \varnothing\\
&\qquad x \in X_1.
\end{aligned}
$$

Let α_2 be the optimal value and X_2 the set of points where the minimum is attained. If $|X_2| = 1$, it is the nucleolus, otherwise let $\mathcal{B}_2 = \{S \subseteq N \mid e(S,x) = \alpha_2,\ x \in X_2\}$. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a balanced collection, since $\mathcal{D}(\alpha_2, x, v) = \mathcal{B}_1 \cup \mathcal{B}_2$, and so on.

Example 4.1. Let $N = \{1,2,3,4\}$, the game defined by (omitting braces and commas)

$$
\begin{aligned}
&v(N) = 100\\
&v(123) = 95, \quad v(124) = 85, \quad v(134) = 80, \quad v(234) = 55\\
&v(\{i,j\}) = 50 \text{ for all } i,j\\
&v(\{i\}) = 0 \text{ for all } i.
\end{aligned}
$$

and consider $X = PI(N,v)$. Applying the above procedure, we find:

$$
\text{Min } \alpha
$$

$$
\text{s.t.}\ \left\{
\begin{array}{ccccccccccccc}
x_1 & + & x_2 & + & x_3 & & & + & \alpha & \geqslant & 95\\
x_1 & + & x_2 & & & + & x_4 & + & \alpha & \geqslant & 85\\
x_1 & & & + & x_3 & + & x_4 & + & \alpha & \geqslant & 80\\
 & & x_2 & + & x_3 & + & x_4 & + & \alpha & \geqslant & 55\\
 & & & & x_i & + & x_j & + & \alpha & \geqslant & 50\\
 & & & & & & x_i & + & \alpha & \geqslant & 0\\
x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 100
\end{array}
\right.
$$

The solution is $\alpha_1 = 10$, and the tight constraints are the 2 first ones, plus $x_3 + x_4 + \alpha = 50$, which yields $\mathcal{B}_1 = \{123, 124, 34\}$ (note that it is a balanced collection). Using the system of the 3 tight constraints, we deduce that X_1 is given by

$$
\left\{
\begin{array}{ccccc}
x_1 & + & x_2 & = & 60\\
x_1 & & & \geqslant & 30\\
 & & x_2 & \geqslant & 25\\
x_3 & & & = & 25\\
x_4 & & & = & 15
\end{array}
\right.
$$

As X_1 is not a singleton, we process one step further. The new LP is:

$$
\begin{aligned}
&\text{Min } \alpha \\
&\text{s.t.} \left\{
\begin{array}{ccccccccccc}
x_1 & & & + & x_3 & + & x_4 & + & \alpha & \geqslant & 80 \\
 & & x_2 & + & x_3 & + & x_4 & + & \alpha & \geqslant & 55 \\
x_1 & & & + & x_3 & & & + & \alpha & \geqslant & 50 \\
x_1 & & & & & + & x_4 & + & \alpha & \geqslant & 50 \\
 & & x_2 & + & x_3 & & & + & \alpha & \geqslant & 50 \\
 & & x_2 & & & + & x_4 & + & \alpha & \geqslant & 50 \\
 & & & & & & x_i & + & \alpha & \geqslant & 0 \\
 & & & & & & & & x & \in & X_1
\end{array}
\right.
\end{aligned}
$$

After simplification, the program reduces to

$$
\begin{aligned}
&\text{Min } \alpha \\
&\text{s.t.} \left\{
\begin{array}{ccccc}
x_1 & + & \alpha & \geqslant & 40 \\
x_2 & + & \alpha & \geqslant & 35 \\
x_1 & + & x_2 & = & 60 \\
x_1 & & & \geqslant & 30 \\
x_2 & & & \geqslant & 25
\end{array}
\right.
\end{aligned}
$$

The solution is $\alpha_2 = 7.5$, with $x_1 = 32.5$ and $x_2 = 27.5$, which is the unique solution. Hence X_2 is a singleton and we find $\mathcal{B}_2 = \{134, 24\}$. One can check that $\mathcal{B}_1 \cup \mathcal{B}_2 = \{123, 124, 134, 24, 34\}$ is a balanced collection.

Chapter 5

Bargaining

5.1 Example and notation

The general situation is the following: two players must share one unit of a divisible good (e.g., a cake). If they reach an agreement (α, β), $\alpha, \beta \geqslant 0$, $\alpha+\beta \leqslant 1$, they divide the good according to (α, β). Otherwise, they receive nothing (or some predefined amount). We suppose that each player has a utility function representing his preference.

Definition 5.1. A *bargaining problem* with 2 players is a pair (S, d), with

(i) $S \subseteq \mathbb{R}^2$ is compact (closed and bounded), convex, and nonempty.

(ii) $d = (d_1, d_2) \in S$ is such that there exists $x \in S$ with $x_1 > d_1$ and $x_2 > d_2$, and is called the *disagreement point*.

Interpretation: If the 2 players reach an agreement $x \in S$, they receive the respective utilities x_1, x_2, otherwise they receive d_1, d_2, respectively.

Example 5.1. Suppose that the utility functions are

$$u_1(\alpha) = \alpha, \quad u_2(\alpha) = \sqrt{\alpha} \qquad (\alpha \in [0, 1]).$$

Then an efficient sharing $(\alpha, 1-\alpha)$ leads to a utility vector $(\alpha, \sqrt{1-\alpha})$. If in case of disagreeement the players receive nothing, the corresponding bargaining problem is defined by

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leqslant x_1, x_2 \leqslant 1, x_2 \leqslant \sqrt{1-x_1}\}, \quad d = (0, 0).$$

Let us denote by $\mathcal{B}$ the set of all bargaining problems.

Definition 5.2. A *solution* of a bargaining problem is a mapping

$$\begin{aligned} F : \mathcal{B} &\to \mathbb{R}^2 \\ (S, d) &\mapsto F(S, d) \in S \end{aligned}$$

5.2 The Nash solution

Nash imposes to F the following 4 properties:

(i) **Weak Pareto optimality (WPO):** $F(S, d)$ belongs to $W(S)$ for every $(S, d) \in \mathcal{B}$, with

$$W(S) = \{x \in S \mid \forall y \in \mathbb{R}^2, y_1 > x_1 \text{ and } y_2 > x_2 \text{ imply } y \notin S\}$$

the set of weakly Pareto optimal solutions.

(ii) We say that (S, d) is a *symmetric* bargaining problem if $d_1 = d_2$ and S is symmetric w.r.t. the main diagonal, i.e., $(x_1, x_2) \in S$ iff $(x_2, x_1) \in S$.

Symmetry (S): $F_1(S, d) = F_2(S, d)$ for all symmetric (S, d).

(iii) **Scale covariance (SC):** for every $(S, d) \in \mathcal{B}$, for every $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ with $a_1, a_2 > 0$

$$F(aS + b, ad + b) = aF(S, d) + b$$

where $aS + b = \{(a_1x_1 + b_1, a_2x_2 + b_2) \mid (x_1, x_2) \in S\}$, and $aF(S, d) = (a_1F_1(S, d), a_2F_2(S, d))$.

(iv) **Independence of Irrelevant Alternatives (IIA):** For every $(S, d), (T, d) \in \mathcal{B}$ with $T \subseteq S$ and $F(S, d) \in T$, we have $F(T, d) = F(S, d)$.

Definition 5.3. The *Nash solution* $N : \mathcal{B} \to \mathbb{R}^2$ is defined by

$$N(S, d) = \operatorname{argmax}\{(x_1 - d_1)(x_2 - d_2) \mid x \in S, x \geqslant d\}.$$

It can be shown that the Nash solution is well defined (exists and is unique).

For the bargaining problem in Example 5.1, we find $F(S, d) = (2/3, \sqrt{1/3})$, since it is the solution of $\max_{0 \leqslant \alpha \leqslant 1} \alpha\sqrt{1-\alpha}$.

Theorem 5.1. (Nash, 1950) $F = N$ iff F satisfies (WPO), (S), (SC) and (IIA).

5.3 The solution of Raiffa-Kalai-Smorodinsky

Kalai and Smorodinsky have replaced the somewhat controversial axiom IIA of Nash by a property called individual monotonicity.

We define the *utopia point* of $(S, d) \in \mathcal{B}$ by

$$u(S, d) = (\max\{x_1 \mid x \in S, x \geqslant d\}, \max\{x_2 \mid x \in S, x \geqslant d\}).$$

Individual Monotonicity (IM): For every $(S, d) \in \mathcal{B}$, $T \supseteq S$, such that $u_i(S, d) = u_i(T, d)$ for some $i \in \{1, 2\}$,

$$F_j(S, d) \leqslant F_j(T, d) \text{ for } j \neq i.$$

The *Raiffa-Kalai-Smorodinsky solution* $R(S, d)$ is defined as the intersection of the frontier of S with the segment (d, u).

Theorem 5.2. $F = R$ iff F satisfies (WPO), (S), (SC) and (IM).

5.4 The egalitarian solution

Monotonicity (M): $F(S,d) \leqslant F(T,d)$ for all $(S,d) \in \mathcal{B}$ and $T \supseteq S$.

We say that (S,d) is *comprehensive* if for every $z, x \in S$, $x \leqslant y \leqslant z$ implies $y \in S$. Let us denote by $\mathcal{B}^c$ the subclass of comprehensive bargaining problems.

The *egalitarian solution* $E : \mathcal{B}^c \to \mathbb{R}^2$ is defined by

$$E(S,d) \in W(S) \text{ is such that } E_1(S,d) - d_1 = E_2(S,d) - d_2,$$

i.e., the excess w.r.t. the disagreement point is equal for the two players. We introduce the following property:

Translation Covariance (TC): For every $(S,d) \in \mathcal{B}^c$ and $e \in \mathbb{R}^2$,

$$F(S+e, d+e) = F(S,d) + e.$$

Theorem 5.3. Let $F : \mathcal{B}^c \to \mathbb{R}^2$. Then $F = E$ iff F satisfies (WPO), (M), (S) and (TC).

Chapter 6

Bankruptcy problems

6.1 Bankruptcy problems and division rules

Consider an estate $E > 0$, to be shared among a set of agents (claimants) N. Claimant i has demand $c_i > 0$, and we set $c = (c_1, \ldots, c_n)$.

We call (c, E) a *bankruptcy problem* if $\sum_{i \in N} c_i \geqslant E$. We denote by $\mathcal{C}^N$ the set of all bankruptcy problems.

Other interpretation: A bankruptcy problem can be seen as a *taxation problem*. Agents in N are the tax payers, c_i is the income of agent i, and E is the cost of some common project.

A *division rule* R is a mapping $(c, E) \mapsto R(c, E) = x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = E$ and $0 \leqslant x \leqslant c$.

6.2 Main division rules

We start by giving two famous examples of bankruptcy problems from the Talmud.

> **The contested garment.** *Two men contest the property of a garment, whose value is 200. The first man claims half of it (100), and the second one claims the totality of it (200). The Talmud recommends the solution* $(50, 150)$.
>
> **The estate division problem.** *A man has 3 wives, and the marriage contract stipulates that at his death, they will receive 100, 200 and 300, respectively. The man dies, and it is discovered that his estate is only 100. The Talmud recommends as solution* $(33^1/_3, 33^1/_3, 33^1/_3)$. *If the estate is 300, it recommends* $(50, 100, 150)$*, and if the estate is 200, it recommends* $(50, 75, 75)$.
>
> N.B.: None of the following rules, except the Talmud rule, is able to explain the figures in both examples.

A simple solution for the contested garment problem. Let us call i and j the two claimants. i claims c_i means that $E - c_i$ is conceded to j if this amounts is nonnegative, otherwise it is 0 (and similarly for j). Hence, each of them takes what the other one is conceding to him, and the rest is divided in two equal parts. This leads to:

"Concede and Divide" rule (CD), for $n = 2$ agents.

$$\mathrm{CD}_i(c, E) = \underbrace{\max(E - c_j, 0)}_{\text{conceded part}} + \underbrace{\frac{1}{2}\Big(E - \sum_k \max(E - c_k, 0)\Big)}_{\text{contested part}}$$

This rule is able to explain the contested garment problem, but seems difficult to generalize for more than 2 players.

The Proportional Rule (P).

$$\mathrm{P}(c, E) = \lambda c, \text{ with } \lambda \text{ such that } \sum_i \lambda c_i = E.$$

Variant: c_i is replaced by $\min(c_i, E)$ (truncated proportional rule).

Constrained Equal Awards Rule (CEA). It gives to everybody the same amount, except if this exceeds the claim.

$$\mathrm{CEA}_i(c, E) = \min(c_i, \lambda), \text{ with } \lambda \text{ such that } \sum_j \min(c_j, \lambda) = E.$$

Piniles Rule (Piniles, 1861).

$$\Pi_i(c, E) = \begin{cases} \mathrm{CEA}_i\left(\frac{c}{2}, E\right), & \text{if } \sum_j \frac{c_j}{2} \geqslant E \\ \frac{c_i}{2} + \mathrm{CEA}_i\left(\frac{c}{2}, E - \sum_j \frac{c_j}{2}\right), & \text{otherwise.} \end{cases}$$

This rule is able to explain the figures for the estate division problem, but not of the contested garment problem.

Constrained Egalitarian Rule (CE).

$$\mathrm{CE}_i(c, E) = \begin{cases} \min\left(\frac{c_i}{2}, \lambda\right), & \text{if } \sum_j \frac{c_j}{2} \geqslant E \\ \max\left(\frac{c_i}{2}, \min(c_i, \lambda)\right), & \text{otherwise} \end{cases}$$

with λ such that $\sum_i \mathrm{CE}_i(c, E) = E$.

Constrained Equal Losses Rule (CEL).

$$\mathrm{CEL}_i(c, E) = \max(0, \underbrace{c_i - \lambda}_{\text{loss = what he does not receive}}), \text{ with } \lambda \text{ such that } \sum_j \max(0, c_j - \lambda) = E.$$

This yields the correct figures for the contested garment problem.

The Talmud Rule (T).

$$T_i(c,E) = \begin{cases} \min\left(\frac{c_i}{2},\lambda\right), & \text{with } \lambda \text{ s.t. } \sum_j \min\left(\frac{c_j}{2},\lambda\right) = E, \text{ if } \sum_j \frac{c_j}{2} \geqslant E \\ c_i - \min\left(\frac{c_i}{2},\lambda\right), & \text{with } \lambda \text{ s.t. } \sum_j \left(c_j - \min\left(\frac{c_j}{2},\lambda\right)\right) = E, \text{ otherwise.} \end{cases}$$

Summary:

(i) If $\sum_j \frac{c_j}{2} = E$, everybody receives $\frac{c_i}{2}$.

(ii) If $\sum_j \frac{c_j}{2} \geqslant E$, one applies CEA with $\frac{c}{2}$.

(iii) If $\sum_j \frac{c_j}{2} \leqslant E$, one applies CEL with $\frac{c}{2}$.

Algorithmic definition, supposing $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$:

(i) We make E vary from 0 to $\sum_j \frac{c_j}{2}$.

- We make E increase from 0 till each agent receives $\frac{c_1}{2}$. Then agent 1 does not receive anything more for a while.
- We continue to make E increase till each agent (except the 1st one) receives $\frac{c_2}{2}$. Then STOP for agent 2.
- ...and so on till arriving at $E = \sum_j \frac{c_j}{2}$, in which case every agent i will have received $\frac{c_i}{2}$.

(ii) We make E vary from $\sum_j c_j$ to $\sum_j \frac{c_j}{2}$.

- If $E = \sum_j c_j$, then every agent i receives c_i.
- The diminution of E is shared equally among the agents till every agent has a loss equal to $\frac{c_1}{2}$. Then agent 1 has no more loss.
- We continue to make E decrease while sharing equally the losses among agents (except agent 1), till the losses are equal to $\frac{c_2}{2}$.
- ...and so on till arriving at $E = \sum_j \frac{c_j}{2}$.

We illustrate the method on both examples.

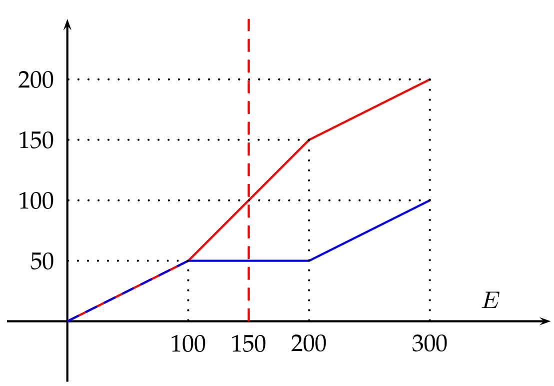


Figure 6.1: The Talmud rule applied to the contested garment problem. $\sum_j c_j/2 = 150$. Blue solid line: x_1, red solid line: x_2

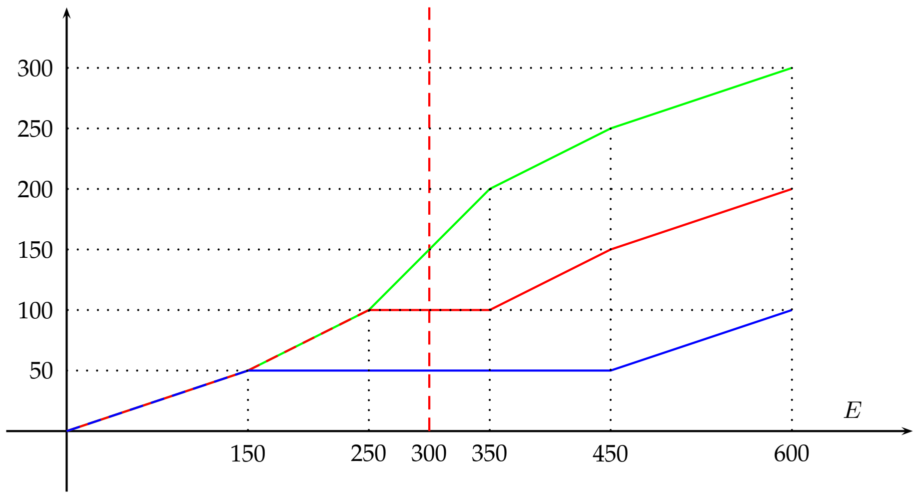


Figure 6.2: The Talmud rule applied to the estate division problem. $\sum_j c_j/2 = 300$. Blue solid line: x_1, red solid line: x_2, green solid line: x_3

The Random Arrival Rule (RA).

$$\mathrm{RA}_i(c,E) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}(N)} \min\Big\{c_i, \max\Big\{E - \sum_{\substack{j\in N\\ \sigma(j)<\sigma(i)}} c_j, 0\Big\}\Big\}.$$

6.3 Relations with solutions of cooperative games

6.3.1 Bargaining solutions

To each bankruptcy problem (c, E), we assign a bargaining problem whose set of feasible solutions is

$$B(c,E) = \{x \in \mathbb{R}^N \mid \sum_i x_i \leqslant E, 0 \leqslant x \leqslant c\}$$

and disagreement point is 0. We say that a division rule corresponds to a bargaining solution if the recommended vector is the same.

Theorem 6.1. (Dagan and Volij 1993) The following correspondences exist:

(i) CEA (Constrained Equal Awards) with the Nash solution

(ii) P (Proportional) with the weighted Nash solution, defined by $\max_{x\geqslant d}\prod_i (x_i - d_i)^{w_i}$, taking $w_i = c_i$ for all i.

(iii) The truncated proportional rule with Kalai-Smorodinsky.

6.3.2 Solutions of coalitional games

We associate to a bankruptcy problem (c, E) the TU-game

$$v(c,E)(S) = \max\Big(\underbrace{E - \sum_{i \in N \setminus S} c_i}_{\text{what remains after agents in } N \setminus S \text{ are served}}, 0\Big) \qquad (S \subseteq N, S \neq \varnothing)$$

Properties of $v(c, E)$:

(i) $v(c, E)$ is convex

(ii) The core of $v(c, E)$ is the set of all possible sharing vectors x, i.e., satisfying $x(N) = E$ and $0 \leqslant x \leqslant c$. Indeed, as for the second condition, for any core element x:

- $x_i \geqslant v(c, E)(\{i\}) \geqslant 0$.
- Supposing $E \geqslant \sum_{i \in N \setminus S} c_i$, we have

$$\begin{aligned} x(S) \geqslant E - c(N \setminus S) &\Leftrightarrow \\ E - x(N \setminus S) \geqslant E - c(N \setminus S) &\Leftrightarrow \\ x(N \setminus S) \leqslant c(N \setminus S) & \end{aligned}$$

which yields for $S = N \setminus i$: $x_i \leqslant c_i$.

Case $n = 2$. The standard solution $x_i = v(c,E)(\{i\}) + \frac{1}{2}\Big(v(c,E)(\{i,j\}) - v(c,E)(\{i\}) - v(c,E)(\{j\})\Big)$ yields

$$x_i = \max(E - c_j, 0) + \frac{1}{2}\Big(E - \sum_k \max(E - c_k, 0)\Big)$$

which is exactly the "Concede and Divide" rule.

General case.

Theorem 6.2. The following correspondences exist:

(i) The Random Arrival rule with the Shapley value

(ii) The Talmud rule with the prenucleolus

(iii) The Constrained Equal Awards rule with the Dutta-Ray solution.

The *Dutta-Ray solution* is the core element which is Lorenz-maximal. We say that x dominates y *in the Lorenz sense* if, letting $x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}$ and $y_{\tau(1)} \leqslant \cdots \leqslant y_{\tau(n)}$, we have

$$\sum_{i=1}^{j} x_{\sigma(i)} \geqslant \sum_{i=1}^{j} y_{\tau(i)} \qquad (j = 1, \ldots, n)$$

and $x(N) = y(N)$.